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**ANALYSIS AND DESIGN  
OF SPACE VEHICLE  
FLIGHT CONTROL SYSTEMS**

**VOLUME V - SENSITIVITY THEORY**

*by Arthur L. Greensite*

*Prepared by*  
**GENERAL DYNAMICS CORPORATION**  
San Diego, Calif.  
*for George C. Marshall Space Flight Center*

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ANALYSIS AND DESIGN OF SPACE VEHICLE  
FLIGHT CONTROL SYSTEMS

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## FOREWORD

This report was prepared under NASA Contract NAS 8-11494 and is one of a series intended to illustrate methods used for the design and analysis of space vehicle flight control systems. Below is a complete list of the reports in the series:

Volume I	Short Period Dynamics
Volume II	Trajectory Equations
Volume III	Linear Systems
Volume IV	Nonlinear Systems
Volume V	Sensitivity Theory
Volume VI	Stochastic Effects
Volume VII	Attitude Control During Launch
Volume VIII	Rendezvous and Docking
Volume IX	Optimization Methods
Volume X	Man in the Loop
Volume XI	Component Dynamics
Volume XII	Attitude Control in Space
Volume XIII	Adaptive Control
Volume XIV	Load Relief
Volume XV	Elastic Body Equations
Volume XVI	Abort

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## 1. STATEMENT OF THE PROBLEM

At the risk of belaboring the obvious, we may point out that a system design and its hardware implementation are never completely identical. Design values, when translated into equipment specifications, usually include permissible deviations (tolerances) that reflect the degree of performance considered acceptable. Tolerances that are too "tight" are generally wasteful of money and effort, while those that are too liberal could compromise performance.

In control systems, once nominal values have been established for component gains and time constants, there remains the problem of determining the effect of deviations from them. A brute-force approach consists of what we may euphemistically call a "parameter study," in which all combinations of selected parameter values are examined with respect to resultant system performance. When a large number of parameters are involved, such a task becomes unmanageably large.

A rational approach to the problem is contained in the methods of sensitivity theory. The concept of sensitivity is an old one in control theory; and, indeed, one of the primary virtues of the principle of feedback was that it provided a degree of insensitivity to parameter variations in the forward loop. This idea is expanded upon in Sec. 3.1.1.

The modern theory of sensitivity is concerned with determining the effects of parameter variations on system performance in a broad sense. It embraces the usual frequency-response methods for single-input/single-output systems, as well as state-variable representations and sampled-data, multivariable, and optimal control systems. The fact that significant theoretical contributions still appear is indicative of the formative state of the theory. Nevertheless, the foundation is sufficiently firm to be extremely useful in system design.

An intelligent use of these methods provides: a framework for the rational specification of system component tolerances; and a quantitative evaluation of how the system is affected by parameter deviations.



## 2. STATE OF THE ART

The inherent ability of a feedback system to minimize the effect of parameter deviations has long been recognized. In fact, this reduction in system sensitivity was originally employed as a quantitative measure of the advantages of feedback.<sup>(2)</sup> The measure of sensitivity, as first defined by Bode<sup>(2)</sup> (and expressed in frequency response terms), had remained unchanged for almost a decade. It is still a sound tool for evaluating the performance quality of a feedback control system in the light of sensitivity requirements.

With the emergence of some of the newer ideas in control theory (root-locus, state-variable, sampled-data, multivariable systems, etc.), a need developed for an expanded notion of the classical concept of sensitivity. Studies by Ur,<sup>(3)</sup> Huang,<sup>(4)</sup> McRuer,<sup>(6)</sup> and Rung<sup>(7)</sup> developed measures of sensitivity for the closed-loop poles of a system in terms of open-loop gain and of open-loop poles and zeros. Questions of sensitivity for sampled-data systems were considered by Lindorff<sup>(14)</sup>, and measures of sensitivity for multivariable systems were developed by Cruz.<sup>(13)</sup>

A basic contribution was made by Horowitz,<sup>(15,19)</sup> who showed that sensitivity analyses need not be restricted to small parameter variations. A natural consequence of this result -- namely, that a conventional linear system could be designed to cope with large parameter variations -- inevitably posed the question of whether (in many cases) an adaptive system is really superior to a well-designed conventional system. Two studies<sup>(20,21)</sup> in the literature apply Horowitz's technique to control problems that were thought to be incapable of solution by conventional linear methods.

With the questions of control system sensitivity in virtually definitive form at present, attention is being focused on sensitivity considerations in optimal control systems. Here the basic problem is essentially the following. Given a system for which an optimal control function has been calculated. The control is optimal in the sense that a prescribed function has been minimized (or maximized). The question arises, "How do variations in system parameters affect the performance function?" Very little work of any significance has been done on this problem. Dorato<sup>(29)</sup> proposed a measure of sensitivity for this case in which he also outlined a computational procedure. The numerical difficulties, however, are formidable, and computer solutions are mandatory for any except the most trivial cases. Pagurek<sup>(9,12)</sup> considered the general problem from a more unified point of view and defined a sensitivity function as the derivative of the optimal performance function with respect to the variable system parameter. His treatment has a theoretical elegance that is potentially very useful. However, it also exhibits a formidable computational complexity, and, perhaps equally important, his sensitivity measure is such that a parameter variation may actually improve the performance function. This would appear to be a

semantic and perhaps logical deficiency that is not in harmony with the usual interpretation of a sensitivity function. In this respect, the definition proposed by Rohrer and Sobral<sup>(28)</sup> is superior, in that any system deviations lead to degradation of the performance function.

In short, sensitivity theory for optimal control systems is still in its formative stages and has not yet reached the definitive form that characterizes feedback control systems.

### 3. RECOMMENDED PROCEDURES

#### 3.1 MEASURES OF SENSITIVITY

In general terms, the basic problem is to provide a quantitative measure of the deviation of a system function when the elements that comprise this function vary in some prescribed manner. This degree of dependence, or sensitivity, is conveniently expressed in terms of the ratio of percentage change in the function to percentage change in the parameter; viz.,

$$S_x^y = \frac{\left(\frac{\partial y}{y}\right)}{\left(\frac{\partial x}{x}\right)} = \frac{x}{y} \frac{\partial y}{\partial x} \quad (1)$$

The form of this expression suggests the alternate definition

$$S_x^y = \frac{\partial (\ln y)}{\partial (\ln x)} \quad (2)$$

Historically, this representation has been used to indicate the ability of a feedback loop to decrease the sensitivity of the overall transfer function to variations in the parameters of the open-loop function. However, the basic idea is useful in studying the dependence of any system function on any parameter. This requires a slightly generalized version of the classical concept of sensitivity. In Secs. 3.1.1-3.1.5, various specialized sensitivity functions, together with their respective interpretation and areas of application, are developed.

##### 3.1.1 System Transfer Function

The classical feedback configuration is shown in Fig. 1. Let

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \quad (3)$$

denote the overall system transfer function.

We are interested in ascertaining the change in  $T(s)$  due to small changes in  $G(s)$ . Using the sensitivity function defined by Eq. (1), we have<sup>†</sup>

$$S_G^T = \frac{G}{T} \cdot \frac{\partial T}{\partial G} = \frac{1}{1 + GH} \equiv \frac{1}{1 + L} \quad (4)$$

<sup>†</sup> The argument,  $s$ , will be dropped whenever it is convenient and where no ambiguity results.

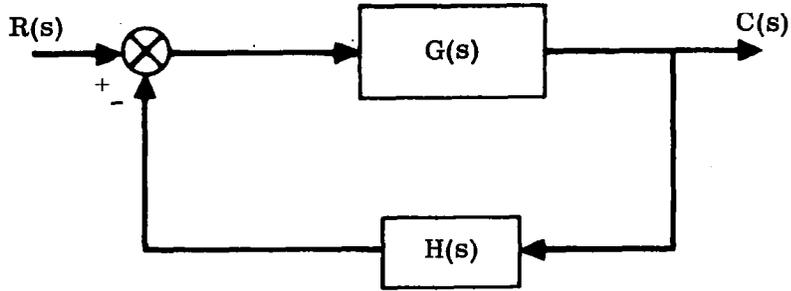


Figure 1. Classical Feedback Configuration

A well-designed system will exhibit a low sensitivity over some predetermined frequency range. Obviously,  $S_G^T \rightarrow 0$  for  $|GH| \rightarrow \infty$  (a great deal of feedback), while  $S_G^T \rightarrow 1$  for  $|GH| \rightarrow 0$ . With no feedback ( $H = 0$ ),  $S_G^T = 1$ .

In the case where  $S_G^T > 1$ , the system without feedback is superior to one with feedback, as far as sensitivity to parameter changes is concerned. In other words, the use of feedback is no assurance in itself that sensitivity to parameter variations is diminished. This idea is intimately related to the fact that over a frequency range of interest, there may indeed be positive feedback. Bode<sup>(2)</sup> has shown that†

$$\int_0^{\infty} \ln [S_G^T(j\omega)] d\omega = 0$$

which means that in any practical system there is as much positive feedback as there is negative feedback. The problem is therefore one of specifying  $G$  and  $H$  such that, in the frequency band of interest, purely negative feedback is obtained.

Consider, for example, the system of Fig. 1 with

$$G(s) = \frac{K}{s(s+a)}, \quad H(s) = 1 \quad (5)$$

and where  $a$  and  $K$  are positive constants. We readily find that

$$S_G^T = \frac{1}{1+G(s)} = \frac{s(s+a)}{s(s+a)+K} \quad (6)$$

† Assuming that the open-loop transfer function has at least two more poles than zeros; or equivalently, if  $|GH| \rightarrow 0$  at greater than 6 db/octave.

In terms of frequency response, we have

$$|S_G^T(j\omega)| = \sqrt{\frac{\omega^2(\omega^2 + a^2)}{K^2 - 2K\omega^2 + \omega^2(\omega^2 + a^2)}} \quad (7)$$

$$\angle S_G^T(j\omega) = \tan^{-1} \frac{a}{-\omega} - \tan^{-1} \frac{a\omega}{(K - \omega^2)} \quad (8)$$

Thus

$$|S_G^T(j\omega)| < 1 \quad \text{for} \quad \omega^2 < \frac{K}{2}$$

$$|S_G^T(j\omega)| > 1 \quad \text{for} \quad \omega^2 > \frac{K}{2}$$

Obviously, the greater the value of  $K$ , the greater the frequency range in which feedback is effective for reducing the sensitivity parameter variations.

Evaluation of  $S_G^T(j\omega)$  is most conveniently accomplished from a Nyquist plot of the open-loop transfer function. From Eq. (4)

$$S_G^T(j\omega) = \frac{1}{1 + L(j\omega)} \quad (9)$$

The vector  $[1 + L(j\omega)]$  is shown in the Nyquist plot of Fig. 2. It is apparent that in this case,  $|1 + L(j\omega)| > 1$  for frequencies less than  $\omega_1$ . This, in turn, means that  $|S_G^T(j\omega)| < 1$  for  $\omega < \omega_1$ . Therefore, a well-designed system, in terms of low sensitivity, will have a bandwidth less than  $\omega_1$ .

If a plot of the closed-loop frequency response is available,  $S_G^T(j\omega)$  may be read directly off the diagram, as shown in Fig. 3.

For computational purposes, if we denote the perturbed values of the open- and closed-loop transfer functions by  $G^*(s)$  and  $T^*(s)$ , then with

$$\Delta G(s) = G^*(s) - G(s) \quad (10)$$

$$\Delta T(s) = T^*(s) - T(s)$$

we have

$$S_G^T \approx \frac{G}{T} \cdot \frac{\Delta T}{\Delta G} \quad (11)$$

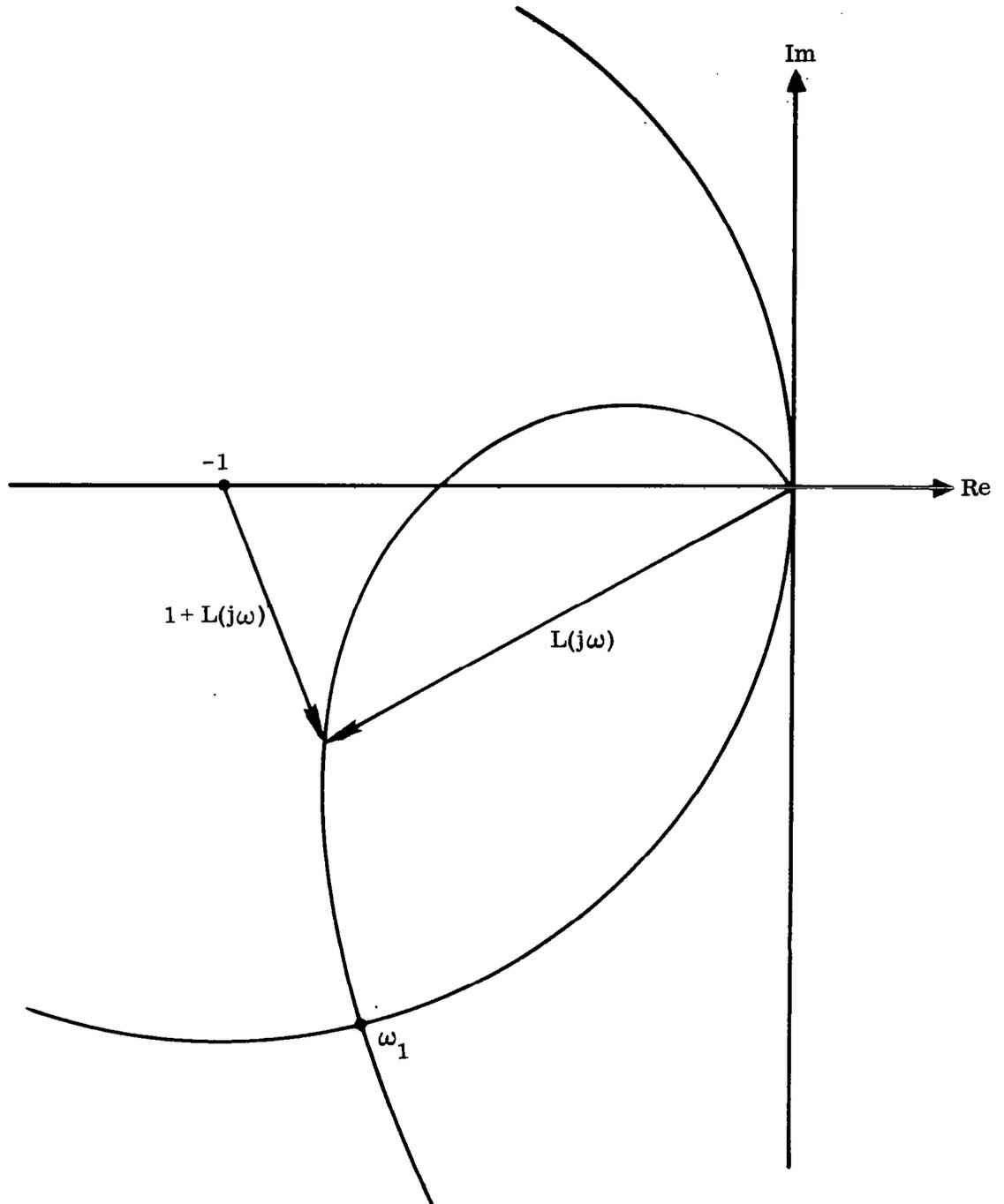


Figure 2. Nyquist Plot of Open-Loop Transfer Function

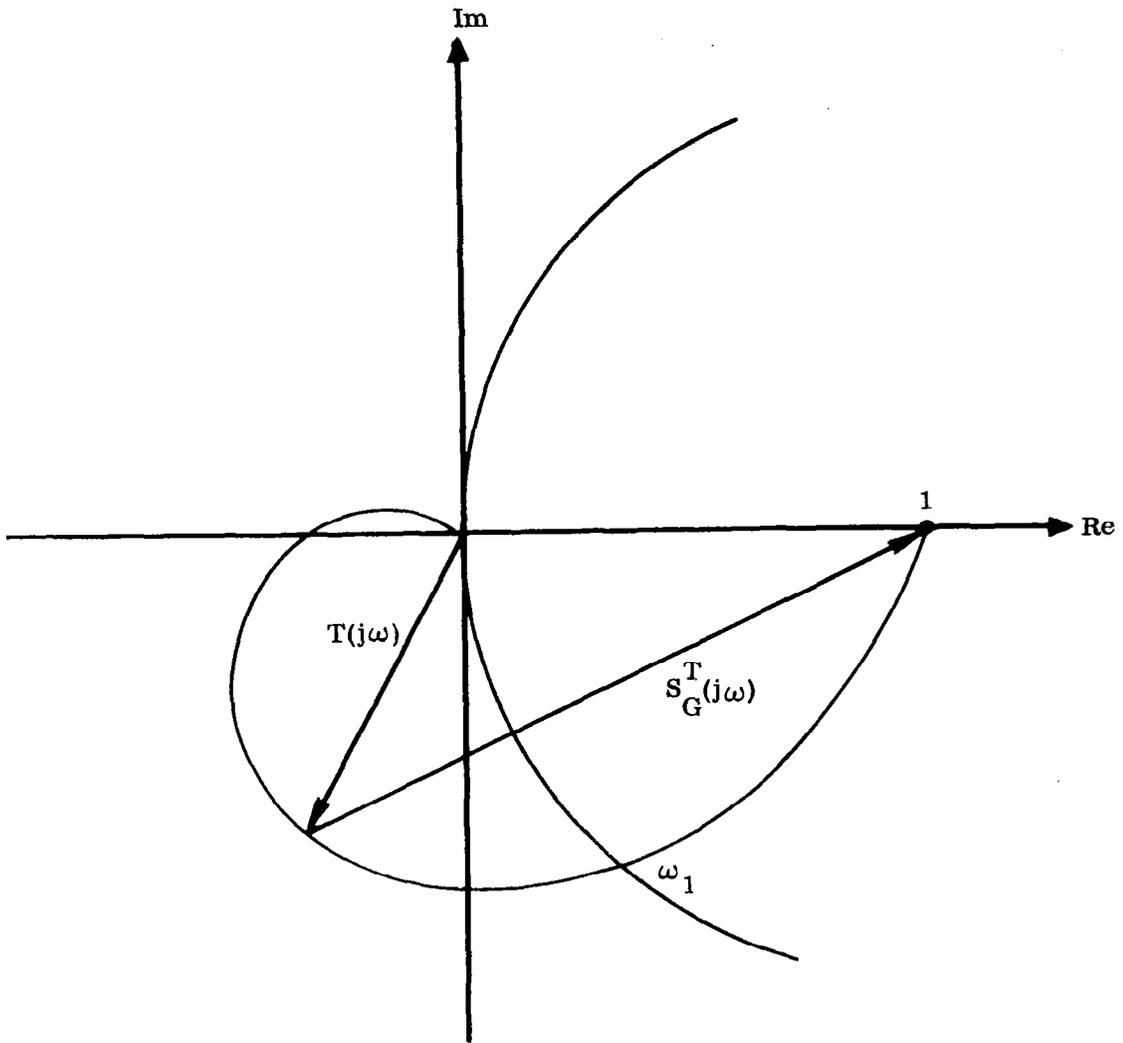


Figure 3. Nyquist Plot of Closed-Loop Transfer Function

so that

$$T^* = T + \Delta T = T \left[ 1 + \frac{\Delta G}{G} S_G^T \right] \quad (12)$$

Consequently, if the nominal and perturbed transfer functions are specified, the sensitivity,  $S_G^T$ , may be used directly to calculate the new transfer function,  $T(s)$ .

The ideas discussed above constitute the classical notions of sensitivity, which, because of historical precedence, were necessarily expressed in terms of frequency response. Modern control theory, which deals with such concepts as root locus, state variables, etc., would therefore seem to require an expanded approach to the definitions and use of sensitivity. This is considered in the following sections.

### 3.1.2 Closed-Loop Poles

Referring to the unity feedback system shown in Fig. 4, we assume that the open-loop transfer function is expressed in terms of its poles and zeros as

$$G(s) = \frac{C(s)}{E(s)} = \frac{K \prod_{j=1}^n (s + z_j)}{\prod_{i=1}^{m+n} (s + p_i)} = \frac{K\alpha(s)}{\beta(s)} \quad (13)$$

A deviation in  $G(s)$  may be due to a variation in: open-loop gain,  $K$ ; the zero,  $z_j$ ; the pole,  $p_i$ ; or any combination of these.

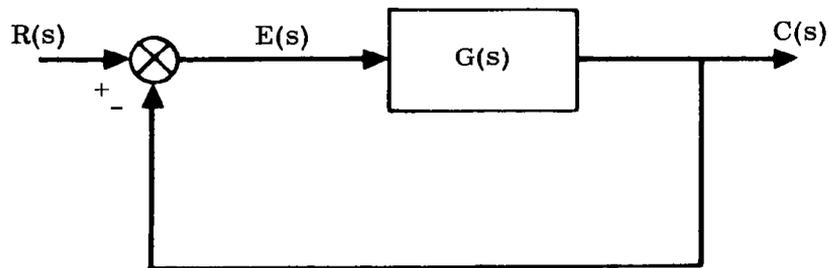


Figure 4. Unity Feedback System

Obviously a change in  $G(s)$  manifests itself in a shift of the closed-loop poles of the system; i. e., a shift in the poles of

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{K\alpha(s)}{\beta(s) + K\alpha(s)} = \frac{K \prod_{j=1}^n (s + z_j)}{\prod_{i=1}^{m+n} (s + q_i)} \quad (14)$$

assuming  $m \geq 1$ .

Conventional root-locus techniques may be used to determine how the closed-loop poles are affected by changes in loop gain only. For present purposes, we seek to determine the sensitivity of the closed-loop poles to variations in  $K$ ,  $z_j$ , and  $p_i$ . This particular sensitivity may be defined in various ways. Ur<sup>(3)</sup> takes

$$S_x^i = x \frac{\partial q_i}{\partial x}$$

where  $x$  may be the open-loop gain, a pole, or a zero. On the other hand, Huang<sup>(4)</sup> uses

$$S_x^i = \frac{x}{q_i} \frac{\partial q_i}{\partial x}$$

which is closest in form to the classical definition of Eq. (1) or (2).

In the ensuing discussion, we will adopt the type defined by McRuer and Stapleford<sup>(6)</sup>; viz.,

$$S_K^i = K \frac{\partial q_i}{\partial K} \quad (15)$$

$$S_{z_j}^i = \frac{\partial q_i}{\partial z_j} \quad (16)$$

$$S_{p_i}^i = \frac{\partial q_i}{\partial p_j} \quad (17)$$

Definitions (15) - (17) are particularly convenient, since (as will be shown subsequently) they satisfy a number of very useful relationships with the open-loop poles and zeros and with the residues of the closed-loop transfer function. These

enable the sensitivity properties of the system to be expressed in a very simple and enlightening form.

We now write the open-loop transfer function as

$$G = G(s, K, z_j, p_j) \quad (18)$$

to emphasize that it is a function of not only  $s$  and  $K$ , but also the pole and zero locations. Forming the total differential, we have

$$dG = \frac{\partial G}{\partial s} ds + \frac{\partial G}{\partial K} dK + \sum_{j=1}^n \frac{\partial G}{\partial z_j} dz_j + \sum_{j=1}^{m+n} \frac{\partial G}{\partial p_j} dp_j \quad (19)$$

Since  $-q_i$  is a root of

$$1 + G(s) = 0 \quad (20)$$

the total differential of Eq. (19) must be zero for  $s = -q_i$ . Setting  $dG = 0$  and  $s = -q_i$  in Eq. (19), we obtain, after rearranging terms.

$$dq_i = \left. \left( \frac{\partial G}{\partial K} \right) \right|_{s=-q_i} dK + \sum_{j=1}^n \left. \left( \frac{\partial G}{\partial z_j} \right) \right|_{s=-q_i} dz_j + \sum_{j=1}^{m+n} \left. \left( \frac{\partial G}{\partial p_j} \right) \right|_{s=-q_i} dp_j \quad (21)$$

Since  $q_i$  is itself a function of  $K$ ,  $z_j$ , and  $p_j$  an alternate expression for  $dq_i$  may be written; viz.,

$$dq_i = q_i(K, z_j, p_j) \quad (22)$$

Taking the total differential,

$$dq_i = K \frac{\partial q_i}{\partial K} \frac{dK}{K} + \sum_{j=1}^n \frac{\partial q_i}{\partial z_j} dz_j + \sum_{j=1}^{m+n} \frac{\partial q_i}{\partial p_j} dp_j \quad (23)$$

By virtue of Eqs. (15) - (17), this may be written as

$$dq_i = S_K^i \frac{dK}{K} + \sum_{j=1}^n S_{z_j}^i dz_j + \sum_{j=1}^{m+n} S_{p_j}^i dp_j \quad (24)$$

After equating like coefficients in Eqs. (21) and (24), we find

$$S_{K}^i = K \frac{\partial q_i}{\partial K} = K \left[ \frac{(\partial G)}{(\partial K)} \right]_{s=-q_i} \quad (25)$$

$$S_{z_j}^i = \frac{\partial q_i}{\partial z_j} = \left[ \frac{(\partial G)}{(\partial z_j)} \right]_{s=-q_i} \quad (26)$$

$$S_{p_j}^i = \frac{\partial q_i}{\partial p_j} = \left[ \frac{(\partial G)}{(\partial p_j)} \right]_{s=-q_i} \quad (27)$$

Note that the gain sensitivity is based on a fractional change in K, while the pole and zero sensitivities are based on absolute shifts in  $p_j$  and  $z_j$ . This apparent lack of harmony in the definitions is of little concern, since these definitions lead to simple and instructive relations for the respective sensitivities.

Recalling that  $G(-q_i) = -1$ , while

$$\frac{\partial G}{\partial K} = \frac{G}{K}$$

$$\frac{\partial G}{\partial z_j} = \frac{G}{(s + z_j)}$$

$$\frac{\partial G}{\partial p_j} = -\frac{G}{(s + p_j)}$$

from Eq. (13), the sensitivity relations, (25) - (27), reduce to

$$S_{K}^i = - \left[ \frac{1}{(\partial G)} \right]_{s=-q_i} \quad (28)$$

$$S_{z_j}^i = \frac{S_K^i}{(z_j - q_i)} \quad (29)$$

$$S_{p_j}^i = \frac{S_K^i}{(q_i - p_j)} \quad (30)$$

If we now let  $Q_i$  denote the residue of  $T(s)$  at the pole,  $-q_i$ , then

$$S_K^i = Q_i \quad (31)$$

This relation is readily proved as follows. From Eq. (14),

$$T(s) = \frac{G(s)}{1 + G(s)} = \sum_{i=1}^{m+n} \frac{Q_i}{(s + q_i)} \quad (32)$$

where<sup>(16)</sup>

$$Q_i = \left[ \frac{(s + q_i) G(s)}{1 + G(s)} \right]_{s=-q_i} \quad (33)$$

assuming that all the poles of  $T(s)$  are simple.

Let

$$\Phi_i(s) = \frac{(s + q_i) G(s)}{1 + G(s)} \quad (34)$$

Then by definition

$$\Phi_i(-q_i) = Q_i \quad (35)$$

Writing Eq. (34) in the form

$$[1 + G(s)] \Phi_i(s) = (s + q_i) G(s)$$

and differentiating both sides with respect to  $s$ ,

$$(1 + G)\Phi'_i + \Phi_i G' = G + (s + q_i)G'$$

where

$$(\ )' \equiv \frac{d}{ds}(\ )$$

Solving for  $\Phi_i$ ,

$$\Phi_i(s) = \frac{G(s) + (s + q_i)G'(s) - [1 + G(s)]\Phi'_i(s)}{G'(s)}$$

But since  $G(-q_i) = -1$ , we obtain

$$\Phi_i(-q_i) = -\left. \frac{1}{G'(s)} \right]_{s=-q_i} = Q_i = S_K^i$$

Q. E. D.

Another important relation satisfied by the pole and zero sensitivities is

$$\sum_{j=1}^n S_{z_j}^i + \sum_{j=1}^{m+n} S_{p_j}^i = 1 \quad (36)$$

This is, in fact, a direct consequence of Eqs. (28) - (30) and the form of  $G(s)$ . We note that

$$\left. \frac{\partial G}{\partial s} \right]_{s=-q_i} = K \left[ \frac{\beta\alpha' - \alpha\beta'}{\beta^2} \right]_{s=-q_i} = \left[ G \left( \frac{\alpha'}{\alpha} - \frac{\beta'}{\beta} \right) \right]_{s=-q_i} = - \left( \frac{\alpha'}{\alpha} - \frac{\beta'}{\beta} \right)_{s=-q_i} \quad (37)$$

But

$$\alpha' = \alpha \sum_{j=1}^n \frac{1}{s + z_j}$$

$$\beta' = \beta \sum_{j=1}^{m+n} \frac{1}{s + p_j}$$

Therefore

$$S_K^i = - \frac{1}{\left(\frac{\partial G}{\partial s}\right)_{s=-q_i}} = \frac{1}{\sum_{j=1}^n \frac{1}{z_j - q_i} + \sum_{j=1}^{m+n} \frac{1}{q_i - p_j}} \quad (38)$$

or, equivalently,

$$\sum_{j=1}^n \frac{S_K^i}{z_j - q_i} + \sum_{j=1}^{m+n} \frac{S_K^i}{q_i - p_j} = 1$$

Using Eqs. (29) and (30) leads to (36).

The gain sensitivity,  $S_K^i$  (which is, in general, a complex number), has a simple physical interpretation in the  $s$  plane. It is a vector, tangent to the root locus at  $s = -q_i$ , and oriented in a sense opposite to increasing  $K$ . This follows from the fact that a root locus is a plot of the roots of

$$1 + G(s) = 0$$

The total derivative is

$$dG = \frac{\partial G}{\partial s} ds + \frac{\partial G}{\partial K} dK$$

But along the locus,  $dG = 0$ . Therefore

$$ds = - \left[ \frac{K \left( \frac{\partial G}{\partial K} \right)}{\left( \frac{\partial G}{\partial s} \right)} \right]_{s=-q_i} \frac{dK}{K} = - \frac{dK}{K} S_K^i \quad (39)$$

Since  $\frac{dK}{K}$  is a real number, the direction of  $ds$  along the locus for positive  $\frac{dK}{K}$  is given by  $-S_K^i$ .

The classical and gain sensitivities are related in a very simple way. From Eq. (4), with  $H(s) = 1$ ,

$$S_G^T = \frac{1}{1+G} = 1 - \frac{G}{1+G}$$

By virtue of Eqs. (31) and (32), this becomes

$$S_G^T = 1 - \sum_{i=1}^{m+n} \frac{S_K^i}{(s + q_i)} \quad (40)$$

Thus the classical sensitivity is interpreted as a weighted sum of the gain sensitivities.

The basic problem of this section is the determination of the shift of the closed-loop poles for prescribed (small) variations in gain and open-loop poles and zeros. In view of the foregoing discussions, the main relation is given by

$$dq_i = S_K^i \left[ \frac{dK}{K} + \sum_{j=1}^n \frac{dz_j}{(z_j - q_i)} + \sum_{i=1}^{m+n} \frac{dp_j}{(q_i - p_j)} \right] \quad (41)$$

which is obtained by combining Eqs. (24), (29), and (30).

In a given situation, the nominal values of  $K$ ,  $q_i$ ,  $z_j$ , and  $p_j$  are known. When the variation of  $G(s)$  is expressed in terms of  $dK$ ,  $dz_j$ , and  $dp_j$ , one may determine  $dq_i$  if the gain sensitivity,  $S_K^i$ , is known. Various methods of calculating this quantity are considered next.

### 3.1.2.1 Calculation of Gain Sensitivity

The most accurate calculation of the gain sensitivity is by the Direct Method; viz.,

$$S_K^i = - \left[ \frac{\beta(s)}{\beta'(s) + K\alpha'(s)} \right]_{s=-q_i} \quad (42)$$

This relation follows directly from Eqs. (13) and (28), making use of the fact that  $G(-q_i) = -1$ . For high-order systems, the evaluation of  $S_K^i$  via Eq. (42) becomes quite laborious. One may then use a graphical approach (slightly less accurate), the simplest of which is the so-called Gain Perturbation Method. Here we make use of the incremental approximation to Eq. (15); i. e.,

$$S_K^i \approx K \frac{\Delta q_i}{\Delta K} \quad (43)$$

To employ this relation, it is necessary to have available the location of two closed-loop poles for two slightly different values of open-loop gain. The situation is depicted in Fig. 5. Having  $K$  and  $\Delta K$ , we measure the magnitude of  $\Delta q_i$  directly off

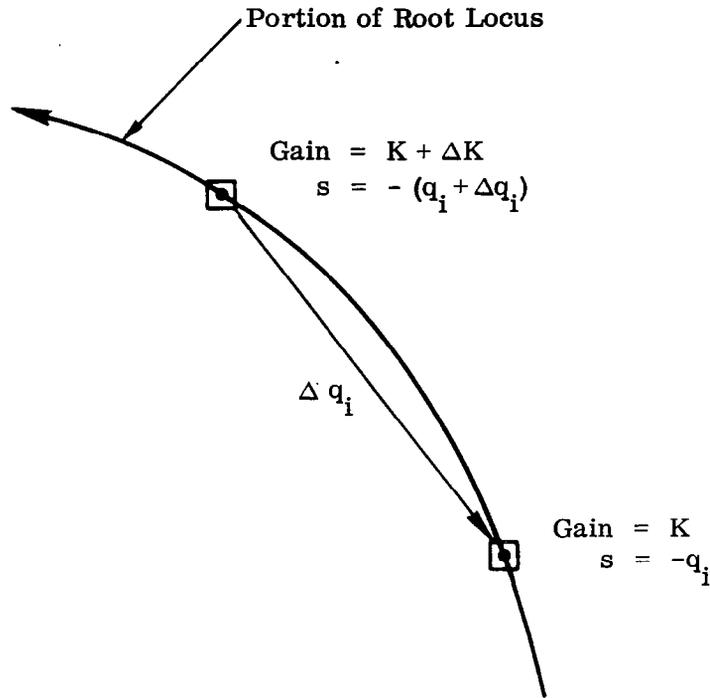


Figure 5. Calculation of Gain Sensitivity by Gain Perturbation Method

the figure. Obviously, since  $K$  and  $\Delta K$  are real numbers, the phase angle associated with  $S_K^i$  is the same as the phase angle of  $\Delta q_i$  (also measured off the figure).

One may devise a more accurate graphical method that is a direct consequence of the fact that the residue of a function of a complex variable at a given singularity can be expressed in terms of vectors to the poles and zeros of the function in the  $s$  plane. Since the gain sensitivity at  $-q_i$  is indeed the residue of  $T(s)$  at the pole  $-q_i$ , we have, by virtue of Eqs. (31) and (33),

$$S_K^i = \left[ \frac{(s + q_i) G(s)}{1 + G(s)} \right]_{s=-q_i}$$

Using Eq. (14), this becomes

$$S_K^i = \frac{K \prod_{j=1}^n (z_j - q_i)}{m+n \prod_{\substack{j=1 \\ j \neq i}} (q_j - q_i)} \quad (44)$$

We may call the approach using Eq. (44) the Vector Method. To use this, it is necessary to have a complete set of closed-loop poles. The gain sensitivity is then expressed in terms of a product of vectors from the open-loop zeros divided by the product of vectors from the closed-loop poles. The accuracy is limited only by the accuracy of the graphical plot.

At this point, it is instructive to consider the application of the above ideas in a specific case.

Example 1: The schematic of the system is shown in Fig. 4. We take

$$G(s) = \frac{K(s+5)}{(s+1)(s+2.75)} = \frac{K\alpha(s)}{\beta(s)} \quad (45)$$

The corresponding root locus is depicted in Fig. 6, which also shows the closed-loop poles corresponding to  $K = 7.25$ . It is required to determine the shift in these closed-loop poles due to prescribed variations in gain and open-loop poles and zero.

Applying Eq. (41) to this case, we have

$$dq_1 = S_K^1 \left[ \frac{dK}{K} + \frac{dz_1}{(z_1 - q_1)} + \frac{\Delta p_1}{(q_1 - p_1)} + \frac{\Delta p_2}{(q_1 - p_2)} \right] \quad (46)$$

By direct measurement of the vector quantities involved (see Fig. 7),

$$z_1 - q_1 = 3.00 \angle 100^\circ$$

$$q_1 - p_1 = 5.38 \angle -33^\circ$$

$$q_1 - p_2 = 4.03 \angle -47^\circ$$

$$-q_1 = -5.5 + j2.96$$

It remains to determine  $S_K^1$ . This will be calculated by each of the three methods described in the previous section.

Using the Direct Method, we have, from Eqs. (42) and (45),

$$S_K^1 = - \left[ \frac{\beta}{\beta' + K\alpha} \right]_{s=-q_1} = - \left[ \frac{s^2 + 3.75s + 2.75}{2s + 3.75 + 7.25} \right]_{s=-5.5+j2.96} = 3.68 \angle 10^\circ$$

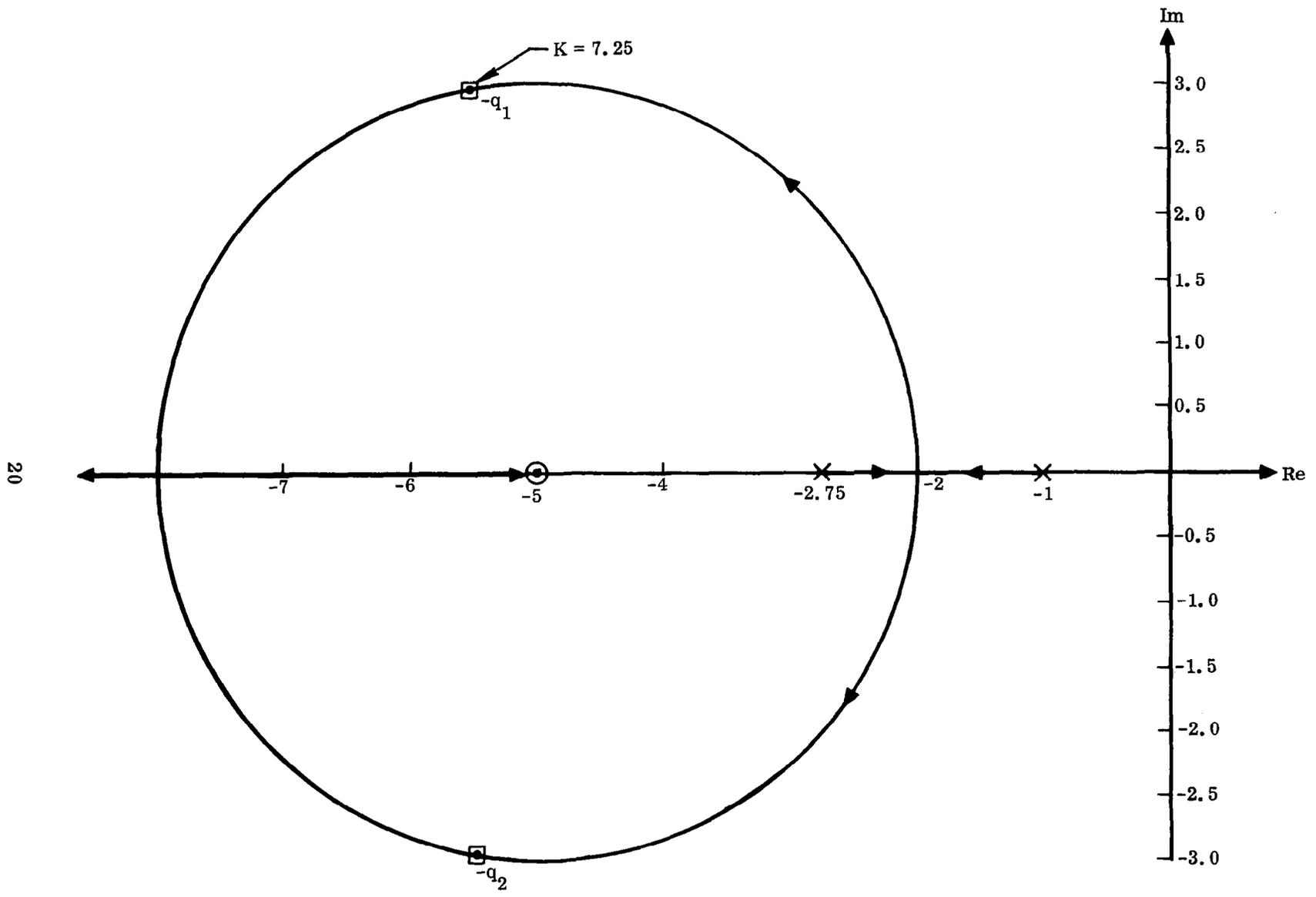


Figure 6. Root Locus for Example 1

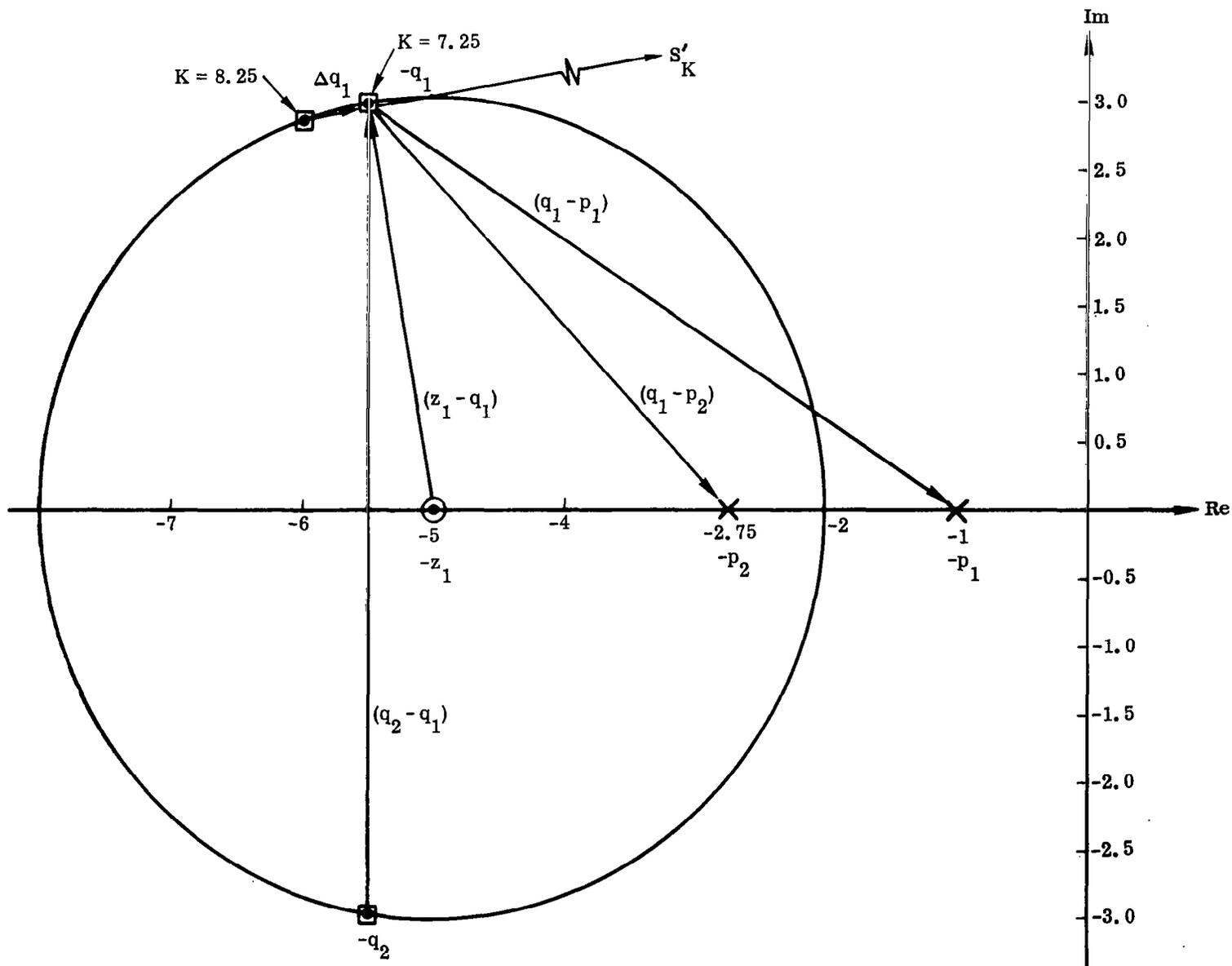


Figure 7. Vector Quantities for Determination of Sensitivity

By the Gain Perturbation Method via Eq. (43),

$$S_K^1 \approx \frac{K \Delta q_1}{\Delta K} = \frac{7.25 \times 0.5 \angle 13^\circ}{1} = 3.62 \angle 13^\circ$$

Finally, using the Vector Method and Eq. (44),

$$S_K^1 = \frac{K(z_1 - q_1)}{(q_2 - q_1)} = \frac{7.25 \times 3.0 \angle 100^\circ}{5.92 \angle 90^\circ} = 3.67 \angle 10^\circ$$

As expected, when the root locus is drawn accurately, the Vector Method yields a value virtually identical to that obtained by the Direct Method.

Assume now that

$$\Delta K = 1$$

$$\Delta p_1 = 0.25$$

$$\Delta p_2 = -0.5$$

$$\Delta z_1 = 0.75$$

Then, from Eq. (46),

$$\begin{aligned} dq_1 &= 3.68 \angle 10^\circ \left[ \frac{1}{7.25} \right. \\ &\quad + \frac{0.75}{3.0 \angle 100^\circ} + \frac{0.25}{5.38 \angle -33^\circ} \\ &\quad \left. - \frac{0.5}{4.03 \angle -47^\circ} \right] \\ &= (3.68 \angle 10^\circ)(0.3154 \angle -81^\circ) \\ &= 1.16 \angle -71^\circ \end{aligned}$$

The new closed-loop pole,  $-q_1^*$ , is then obtained by vector addition as shown in Fig. 8.

As a check on the method, we calculate the closed-loop poles for

$$G(s) = \frac{8.25(s + 5.75)}{(s + 1.25)(s + 2.25)}$$

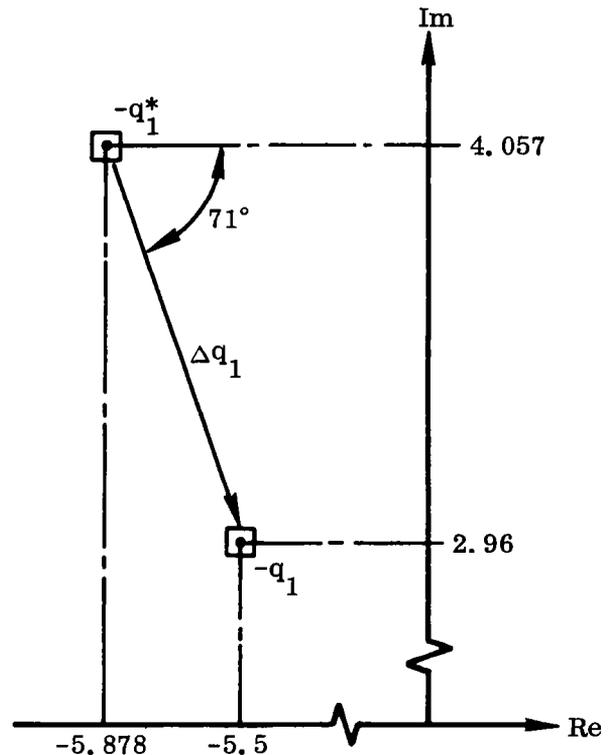


Figure 8. Shift of Closed-Loop Pole in Example 1

We find

$$-q_1^* = -5.875 + j3.968$$

In view of the fact that the parameter deviations were not at all infinitesimal, the agreement is excellent.

Remark: One of the main virtues of the method herein described is that useful qualitative sensitivity features may be obtained with little effort. An examination of Eq. (41) indicates that the sensitivity of a closed-loop pole to variations in a particular open-loop pole or zero diminishes with increasing distance between the two. Thus variations in open-loop poles or zeros far removed from the closed-loop pole in question have a minor influence on the latter. This is perhaps intuitively evident, but Eq. (41) expresses this condition in precise fashion.

### 3.1.3 Eigenvalues and Eigenvectors

We consider a linear stationary system expressed in the state variable format<sup>(17)</sup> as follows.

$$\dot{x} = Ax + w \tag{47}$$

$$x(0) = c \tag{48}$$

where  $x$  is an  $n$ -dimensional state vector ( $n \times 1$  matrix),  $A$  is a constant  $n \times n$  matrix whose typical element will be denoted by  $a_{ij}$ , and  $w$  is a vector forcing function.

It is known that the solution for the system, (47) and (48), is given by<sup>(17)</sup>

$$x = e^{At}c + \int_0^t e^{A(t-\sigma)}w(\sigma)d\sigma \tag{49}$$

where  $e^{At}$  is the transition matrix, which may be computed in several ways. The usual representation is<sup>(17)</sup>

$$e^{At} = M \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & \dots & \dots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} M^{-1} \tag{50}$$

Here the  $\lambda_i$  denote the eigenvalues † of the matrix A, and M is the modal matrix for A.

Thus the distinctive feature of the state-variable representation is that the response is governed by the eigenvalues of the system matrix, A. Using an alternate representation ‡ for the transition matrix,

$$e^{At} = \sum_{i=1}^n e^{\lambda_i t} u_i v_i \quad (51)$$

where

$u_i \equiv$  eigenvector ( $n \times 1$  matrix) corresponding to the eigenvalue,  $\lambda_i$  ( $u_i$  is the  $i^{\text{th}}$  column of the modal matrix M)

$v_i \equiv$  row vector ( $1 \times n$  matrix) whose elements are the  $i^{\text{th}}$  row of  $M^{-1}$

the response is expressed as a weighted sum of the system eigenvectors (modes), which clearly indicates the relative contribution of each mode to the total response.

Now the deviation of any system parameter from its nominal value is reflected in a change in one or more of the elements of the system matrix, A. We investigate the problem of determining the change in the eigenvalue,  $\lambda_j$  (or eigenvector,  $u_j$ ) due to a change in an element  $a_{k\ell}$  of the matrix, A. One measure of sensitivity is the ratio of the (small) increment in  $\lambda_j$  to the (small) increment in  $a_{k\ell}$ . We therefore define the eigenvalue sensitivity

$$S_{k\ell}^j \equiv \frac{\partial \lambda_j}{\partial a_{k\ell}} \quad (52)$$

The problem is now one of finding a suitable expression for  $S_{k\ell}^j$  in terms of given system parameters. Following Laughton, <sup>(11)</sup> we let

$A^* \equiv$  the matrix obtained by replacing the  $(k\ell)^{\text{th}}$  element in A by  $a_{k\ell} + \Delta a_{k\ell}$

$F_{k\ell} \equiv$  the cofactor of the  $(k\ell)^{\text{th}}$  element of  $(A - \lambda I)$

† Throughout the ensuing discussions, these eigenvalues are assumed to be distinct.

‡ See Appendix A.

Then, expanding the determinant of  $(A^* - \lambda I)$  by the elements and cofactors of the  $k^{\text{th}}$  row, we find

$$\begin{aligned} \det [A^* - \lambda I] &= a_{k1} F_{k1}(\lambda) + a_{k2} F_{k2}(\lambda) + \dots \\ &\dots + (a_{kl} + \Delta a_{kl}) F_{kl}(\lambda) + \dots + a_{kn} F_{kn}(\lambda) \\ &= \det [A - \lambda I] + \Delta a_{kl} F_{kl}(\lambda) \end{aligned} \quad (53)$$

Thus the eigenvalues of  $A^*$  are given by the roots of

$$\det [A - \lambda I] + \Delta a_{kl} F_{kl}(\lambda) = 0 \quad (54)$$

We may therefore write Eq. (53) in the form

$$\prod_{i=1}^n (\lambda_i^* - \lambda) = \prod_{i=1}^n (\lambda_i - \lambda) + \Delta a_{kl} F_{kl}(\lambda) \quad (55)$$

where  $\lambda_i^*$  denotes the  $i^{\text{th}}$  eigenvalue (assumed distinct) of  $A^*$ .

Replacing  $\lambda$  by  $\lambda_j$ , Eq. (55) reduces to

$$\frac{\Delta \lambda_j}{\Delta a_{kl}} = \frac{F_{kl}(\lambda_j)}{\prod_{\substack{i=1 \\ i \neq j}}^n (\lambda_i^* - \lambda_j)} \quad (56)$$

where

$$\Delta \lambda_j = \lambda_j^* - \lambda_j \quad (57)$$

Therefore, in the limit

$$S_{kl}^j \equiv \frac{\partial \lambda_j}{\partial a_{kl}} = \frac{F_{kl}(\lambda_j)}{F^{(1)}(\lambda_j)} \quad (58)$$

$$F^{(1)}(\lambda_j) = \left. \frac{d}{d\lambda} \left\{ \det [A - \lambda I] \right\} \right|_{\lambda=\lambda_j} \quad (59)$$

Eq. (58) is an explicit relation for the eigenvalue sensitivity,  $S_{k\ell}^j$ . It is possible to obtain a simpler and more elegant expression as follows. For this purpose, we adopt the notation

$S^j \equiv$  the matrix whose  $(k\ell)$ th element is  $\frac{\partial \lambda_j}{\partial a_{k\ell}}$

$$L(\lambda) \equiv \text{adj}[A - \lambda I]$$

Then the matrix equivalent of Eq. (58) is

$$S^j = \frac{L^T(\lambda_j)}{F^{(1)}(\lambda_j)} \quad (60)$$

We have also<sup>†</sup>

$$L(\lambda_j) = u_j v_j \quad (61)$$

$$F^{(1)}(\lambda_j) = v_j u_j \quad (62)$$

where  $u_j$  and  $v_j$  are the eigenvector and the row vector previously defined. Using this and Eq. (A5) of Appendix A, we find that the eigenvalue sensitivity matrix of Eq. (60) may be expressed as

$$S^j = v_j^T u_j^T \quad (63)$$

This is the basic result of the analysis. Eq. (63) shows that a knowledge of the eigenvalues and modal matrix for  $A$  yields all the information concerning the system sensitivity, in addition to that necessary for determining the system response.

In most practical situations, one system parameter usually appears in several elements of the matrix,  $A$ . The variation of the  $j^{\text{th}}$  eigenvalue is then given by

$$\Delta \lambda_j = \sum_{k, \ell} S_{k\ell}^j \Delta a_{k\ell} \quad (64)$$

where the summation is taken over all perturbed values,  $a_{k\ell}$ .

<sup>†</sup> These relations are derived in Chap. III of Ref. 18.

Proceeding in a completely analogous manner, we find, for the eigenvector sensitivity,

$$S_{k\ell}^{u_j} \equiv \frac{\partial u_j}{\partial a_{k\ell}} = \sum_{\substack{i=1 \\ i \neq j}}^n h_{k\ell}^i u_i \quad (65)$$

where  $h_{k\ell}^i$  is the  $(k\ell)^{\text{th}}$  element of the matrix

$$H^i = \frac{v_i^T u_j^T}{(\lambda_j - \lambda_i)} \quad (66)$$

As pointed out by Laughton<sup>(11)</sup>, the above ideas may be profitably applied to the evaluation of analog computer simulations when time and amplitude scaling is involved. This is important because sensitivity features, as determined by system parameter changes on the computer simulation, often serve as basic guidelines for design. It will be shown that amplitude-scaling materially alters the eigenvalue sensitivity matrix, while time-scaling has no effect on this sensitivity.

To show this, we define a new independent variable by

$$t = \beta \tau, \quad \beta = \text{positive constant}$$

Then the free motion of Eq. (47) takes the form

$$\frac{dx}{d\tau} = \beta A x$$

whose solution is

$$x = \sum_{i=1}^n e^{\lambda_i \tau} u_i v_i^T c = \sum_{i=1}^n e^{\lambda_i \tau} [S^i]^T c$$

using (A14) and Eq. (63).

The eigenvector sensitivity matrix is thus unaffected by a change in time scale.

Consider now a change in amplitude scale. This may be simply represented by the coordinate transformation

$$x = T y$$

where  $T$  is a diagonal matrix. Thus, instead of

$$\dot{x} = Ax$$

we have

$$\dot{y} = By$$

where

$$B = T^{-1}AT \quad (67)$$

Two matrices,  $A$  and  $B$ , related by a transformation of type (67), are said to be connected by a collineatory transformation<sup>(18)</sup>. It is easy to show that in this case,  $A$  and  $B$  have the same eigenvalues. We have

$$T^{-1}AT - \lambda I = T^{-1}(A - \lambda I)T$$

and therefore

$$\det [T^{-1}AT - \lambda I] = (\det T^{-1}) \cdot (\det [A - \lambda I]) \cdot (\det T) = \det [A - \lambda I]$$

Q.E.D.

But  $A$  may be written as

$$A = \sum_{i=1}^n \lambda_i [S^i]^T \quad (68)$$

by virtue of (A9) and (63).

Consequently, using (67) and the fact that  $A$  and  $B$  have the same eigenvalues, we find

$$B = \sum_{i=1}^n \lambda_i [S_*^i]^T \quad (69)$$

where

$$S_*^i = T^{-1} [S^i]^T T \quad (70)$$

It follows, therefore, that the sensitivity properties of the transformed system (amplitude-scaled computer simulation) may differ markedly from the actual system sensitivity.

Example 2: We consider the system described by

$$\dot{\mathbf{x}} = \mathbf{A} \mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

The eigenvalues are

$$\lambda_1 = 1.0$$

$$\lambda_2 = -1 + j$$

$$\lambda_3 = -1 - j$$

The modal matrix is

$$\mathbf{M} = \begin{bmatrix} 0 & 5 & 5 \\ 1 & -(3 + 4j) & -(3 - 4j) \\ 1 & (2 + j) & (2 - j) \end{bmatrix}$$

and its inverse

$$\mathbf{M}^{-1} = \frac{1}{10} \begin{bmatrix} -2 & 2 & 8 \\ (1 + j) & j & -j \\ (1 - j) & -j & j \end{bmatrix}$$

Therefore

$$u_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad u_2 = \begin{bmatrix} 5 \\ -(3+4j) \\ (2+j) \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 5 \\ -(3-4j) \\ (2-j) \end{bmatrix}$$

and

$$v_1 = \frac{1}{5} \begin{bmatrix} -1 & 1 & 4 \end{bmatrix}$$

$$v_2 = \frac{1}{10} \begin{bmatrix} (1+j) & j & -j \end{bmatrix}$$

$$v_3 = \frac{1}{10} \begin{bmatrix} (1-j) & -j & j \end{bmatrix}$$

Assume now that element  $a_{23}$  in matrix A changes from 1 to 0.2. We are interested in determining the shift in the complex eigenvalue,  $\lambda_2$ .

From Eq. (64),

$$\Delta\lambda_2 = S_{23}^2 \Delta a_{23}$$

Now

$$\Delta a_{23} = -0.8$$

while, by Eq. (63),

$$S^2 = v_2^T u_2^T = \frac{1}{10} \begin{bmatrix} (1+j) \\ j \\ -j \end{bmatrix} \begin{bmatrix} 5 & -(3+4j) & (2+j) \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 5(1+j) & (1-7j) & (1+3j) \\ 5j & (4-3j) & -(1-2j) \\ -5j & -(4-3j) & (1-2j) \end{bmatrix}$$

which means that

$$S_{23}^2 = -\frac{(1 - 2j)}{10}$$

Consequently,

$$\Delta\lambda_2 = -\frac{(1 - 2j)}{10}(-0.8) = 0.08 - 0.16j$$

Thus, using (57), we find that the new value of  $\lambda_2$  is

$$\lambda_2^* = (-1 + j) + (0.08 - 0.16j) = -0.92 + 0.84j$$

It is instructive to compare this with the exact value of  $\lambda_2$  obtained by calculating the eigenvalues of the perturbed matrix

$$A^* = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 0.2 \\ -1 & 0 & 1 \end{bmatrix}$$

After a simple calculation, we find

$$\lambda_1^* = 0.815$$

$$\lambda_2^* = -0.9075 + 0.81j$$

$$\lambda_3^* = -0.9075 - 0.81j$$

which indicates very good agreement.

Remark: The particularly attractive feature of the sensitivity measures developed in this section is that these eigenvalue sensitivity matrices are available almost by inspection after the basic response data (modal matrix, transition matrix, etc.) has been calculated for the given system. It is therefore possible to determine quickly, and with little effort, the shifts in the eigenvalues for prescribed variations in system parameters. The elements of the eigenvalue sensitivity matrices provide quick qualitative measures of how the eigenvalues are affected by specific parameter variations.

### 3.1.4 Sampled-Data Systems

The sensitivity measures considered thus far carry over with simple and obvious modifications to sampled-data systems.

Employing Z transforms, Eq. (4) becomes

$$S_G^T(z) = \frac{1}{1 + L(z)} \quad (71)$$

which may be called the classical sensitivity for sampled-data systems. It may be evaluated in the manner shown in Figs. 2 and 3, except that in the present case, we plot  $L(z)$  with  $z = e^{j\omega\tau}$ , and  $\omega$  runs from zero to  $\Pi$  instead of from zero to infinity. In other words, the usual frequency-response methods for sampled-data systems apply.

Furthermore, the discussion of Sec. 3.1.2 applies directly, except that we deal with the z plane instead of s plane poles and zeros.

It is also easy to show that the eigenvalue sensitivities discussed in Sec. 3.1.3 have a direct equivalent for discrete systems.

Consider, for example, the discrete system represented in state-variable form by

$$x_{k+1} = A x_k \quad (72)$$

$$x_0 = c \equiv \text{initial value vector} \quad (73)$$

$x_k$  is an n-dimensional state vector, and subscript k indicates that this is the value at time  $t = t_k$ . For simplicity, the sampling interval has been normalized to unity. A is a constant  $n \times n$  matrix.

It is known that the solution to the system, (72) and (73), is given by<sup>(17)</sup>

$$x_k = A^k c \quad (74)$$

To put this in a more convenient form, we note that

$$\Lambda = M^{-1} A M \quad (75)$$

where M is the modal matrix for A and

---

$\tau$  = sampling period.

$$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \dots & \\ 0 & & & \lambda_n \end{bmatrix}$$

The quantities,  $\lambda_j$ , denote the (distinct) eigenvalues of A. As in Sec. 3.1.3, we write

$$M = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Then, after noting that

$$A^k = M \Lambda^k M^{-1} \tag{76}$$

we may write Eq. (74) as

$$x_k = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} \lambda_1^k & & & 0 \\ & \lambda_2^k & & \\ & & \dots & \\ 0 & & & \lambda_n^k \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} c$$

which reduces to

$$x_k = \sum_{i=1}^n \lambda_i^k u_i v_i c \tag{77}$$

Defining the eigenvalue sensitivity as

$$S_{k\ell}^j = \frac{\partial \lambda_j}{\partial a_{k\ell}} \tag{78}$$

where  $a_{k\ell}$  is the  $(k\ell)^{\text{th}}$  element of  $A$ , an argument completely analogous to that of Sec. 3.1.3 leads to

$$S^j = v_j^T u_j^T \quad (79)$$

which is completely equivalent to (63).

### 3.1.5 Multivariable Systems

A multivariable feedback control system may be viewed as a generalization of the schematic of Fig. 1 in which  $R(s)$  and  $C(s)$  are vectors, and therefore the quantities  $G(s)$  and  $H(s)$  are matrices of transfer functions (or simply matrix transfer functions). In this case, it is not immediately clear how to formulate a matrix equivalent of the sensitivity function defined by Eq. (4). It is apparent, however, that such a matrix sensitivity function should have two primary characteristics: (a) it should provide a quantitative measure of the sensitivity of the closed-loop system to parameter variations as compared with the sensitivity of the open-loop system to these same variations; and (b) it should reduce to the scalar equation (4) for the single-input/single-output case. It has been shown by Cruz and Perkins that both requirements are exhibited by the matrix relating the output errors due to parameter variations in a feedback system to the output errors due to parameter variations in a corresponding open-loop system. This point of view will be developed in what follows. For this purpose, we consider the open- and closed-loop multivariable systems shown in Figs. 9 and 10. Here

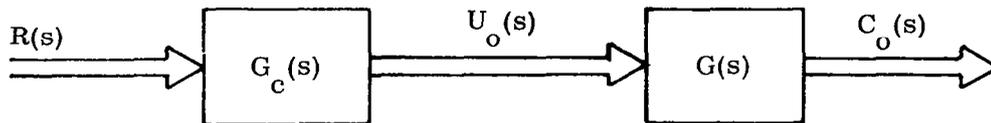


Figure 9. Multivariable Open-Loop Control System

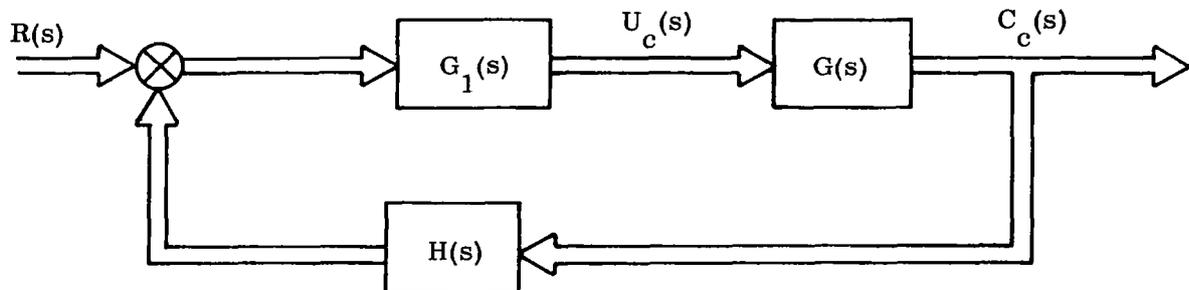


Figure 10. Multivariable Feedback Control System

$R(s) \equiv p$ -dimensional input vector

$U(s) \equiv m$ -dimensional vector

$C(s) \equiv n$ -dimensional output vector

$G_c(s) \equiv m \times p$  matrix

$G_1(s) \equiv m \times p$  matrix

$G(s) \equiv n \times m$  (plant) matrix

$H(s) \equiv p \times n$  matrix

Subscripts  $o$  and  $c$  on vectors  $U(s)$  and  $C(s)$  refer to the vectors in the open- and closed-loop systems respectively. We define also

$$\Delta G(s) = G^*(s) - G(s) \quad (80)$$

where  $G^*(s)$  represents the plant matrix when the parameters differ from nominal.

Referring to Figs. 9 and 10, we find

$$C_o(s) = G(s) U_o(s) \quad (81)$$

$$U_o(s) = G_c(s) R(s) \quad (82)$$

$$C_c(s) = G(s) U_c(s) \quad (83)$$

$$U_c(s) = G_1(s) \left[ R(s) - H(s) C_c(s) \right] \quad (84)$$

If the plant parameters differ from nominal, then we distinguish the vector signals for this case by the starred quantities

$$C_o^*(s) = G^*(s) U_o(s) \quad (85)$$

$$C_c^*(s) = G^*(s) U_c^*(s) \quad (86)$$

$$U_c^*(s) = G_1(s) \left[ R(s) - H(s) C_c^*(s) \right] \quad (87)$$

Note that  $U_o(s)$  remains the same.

Now define the Laplace transform of the vector output errors as follows

$$E_o(s) = C_o(s) - C_o^*(s) \quad (88)$$

$$E_c(s) = C_c(s) - C_c^*(s) \quad (89)$$

It will now be shown that the matrix relating  $E_o(s)$  to  $E_c(s)$  has all the properties of a sensitivity matrix. In this context, it is assumed that  $G_c(s)$ ,  $G_1(s)$ , and  $H(s)$  are such that  $C_c(s) = C_o(s)$  when there are no plant variations.

From (83) and (84), we obtain (dropping the argument  $s$  for simplicity)

$$C_c = (I + G G_1 H)^{-1} G G_1 R \quad (90)$$

Similarly, from (86) and (87),

$$C_c^* = (I + G^* G_1 H)^{-1} G^* G_1 R \quad (91)$$

Substituting (80), (90), and (91) into Eq. (89), we obtain, after some reduction,

$$E_c = (I + G^* G_1 H)^{-1} \left[ \Delta G G_1 (HT - I) \right] R \quad (92)$$

where

$$T = (I + G G_1 H)^{-1} G G_1 \quad (93)$$

Now Eq. (90) can be written as

$$C_c = T R \quad (94)$$

Combining this with (84),

$$U_c = G_1 (I - HT) R \quad (95)$$

But since  $C_c = C_o$  by assumption, we have

$$U_o = U_c = G_1 (I - HT) R \quad (96)$$

Also,

$$E_o = (G - G^*) U_o = -\Delta G U_o = \Delta G G_1 (HT - I)R \quad (97)$$

Substituting this in (92), the latter becomes

$$E_c = (I + G^* G_1 H)^{-1} E_o \quad (98)$$

We define

$$S = (I + G^* G_1 H)^{-1} \quad (99)$$

as the sensitivity matrix for the multivariable system.

Note that for a single input-single output system with small plant variations such that  $G^*$  is approximately equal to  $G$ ,  $S$  is equal to the classical sensitivity as defined by Eq. (4). It will also be observed that while matrices  $G$ ,  $G_1$ ,  $G_c$ , and  $H$  are in general not square, matrix  $S$  is always square.

In certain special cases,  $S$  may be expressed in several interesting ways. For example, if the overall closed-loop transfer matrix is denoted by  $T_c$  with a nominal plant  $G$ , and by  $T_c^*$  with plant  $G^*$ , with analogous meanings for  $T_o$  and  $T_o^*$ , then

$$E_c = C_c - C_c^* = (T_c - T_c^*)R$$

$$E_o = C_o - C_o^* = (T_o - T_o^*)R$$

If  $(T_o - T_o^*)$  is a square nonsingular matrix, we can eliminate  $R$  from the above relations, obtaining

$$E_c = (T_c - T_c^*)(T_o - T_o^*)^{-1} E_o$$

Comparing with (98), we see that

$$S = (T_c - T_c^*)(T_o - T_o^*)^{-1} \quad (100)$$

Now

$$T_o = G G_c \quad (101)$$

and

$$(T_o - T_o^*) = (G - G^*) G_c \quad (102)$$

By virtue of the assumption that the open- and closed-loop systems of Figs. 9 and 10 are equivalent for nominal parameters, we have  $T_c = T_o$ . If it is further assumed that matrices  $G$ ,  $T_o$ ,  $G_c$ , and  $(G - G^*)$  are square and nonsingular, then, from Eqs. (101) and (102),

$$(T_o - T_o^*)^{-1} = G_c^{-1} (G - G^*)^{-1} = T_o^{-1} G (G - G^*)^{-1} = T_c^{-1} G (G - G^*)^{-1}$$

Substituting this in Eq. (100) yields

$$S = \Delta T_c T_c^{-1} G \Delta G^{-1} \quad (103)$$

where

$$\Delta T_c = T_c^* - T_c \quad (104)$$

Note the striking similarity in form between the sensitivity matrix, (103), and the scalar equivalent, (11). It is apparent that the analysis could have proceeded with  $S$  defined by (103) and subsequently interpreted as the matrix relating  $E_c$  to  $E_o$ .

We now seek to provide a criterion that compares the sensitivity of the closed-loop system to that of the open-loop system for prescribed variations in the plant parameters. Following Cruz and Perkins<sup>(13)</sup>, we take, as an index of performance,

$$I.P. = \int_0^{t_f} e^T(t) e(t) dt \quad (105)$$

$$e(t) = \mathcal{L}^{-1}[E(s)]$$

where  $E(s)$  is either  $E_c(s)$  or  $E_o(s)$ . In practical situations,  $t_f$  may be taken as four or five times the largest time constant of the system. Now for the feedback system to be superior to the corresponding open-loop system, the inequality

$$\int_0^{t_f} e_c^T(t) e_c(t) dt < \int_0^{t_f} e_o^T(t) e_o(t) dt \quad (106)$$

must be satisfied. It may be shown<sup>(13)</sup> that this condition is ensured if

$$\left[ S(-j\omega) \right]^T \left[ S(j\omega) \right] < I \quad (107)$$

For the scalar case, it was shown in Sec. 3.1.1 that  $|S(j\omega)| < 1$  was sufficient to ensure that the feedback system has better sensitivity properties than the open-loop system. The matrix equivalent of this is exhibited in (107). For a well-designed system, it is therefore necessary to satisfy (107) over the frequency band of interest.

### 3.2 DESIGN FOR SENSITIVITY

The methods of Sec. 3.1 permit the performance quality of a given system to be evaluated in terms of its sensitivity to parameter variations. This is the analysis problem. It is generally taken as an afterthought, if considered at all. In one sense, this omission is not often serious, since a feedback system has "built-in" sensitivity features that are adequate in most designs. However, one may take the point of view that a prescribed sensitivity is a fundamental design parameter and may determine appropriate compensation networks accordingly. Techniques for accomplishing this are not nearly as well-developed as for conventional compensation (to ensure relative stability, time or frequency response, etc.). One method, useful in certain cases, is described in Sec. 3.2.1.

Another aspect of the sensitivity problem, first pointed out by Horowitz<sup>(19)</sup>, is that in certain instances, one mistakenly treats what is basically a sensitivity problem as one of adaptive control! He introduces the concept of "sensitivity in the large," and shows that in many cases, a suitably compensated "high gain" system exhibits the features generally thought to be obtainable only by adaptive methods. This approach will be discussed in Sec. 3.2.2.

#### 3.2.1 Compensation Networks

One design method for feedback system compensation is based on the assumption that the system-response specifications can be expressed in terms of desired locations of dominant closed-loop roots<sup>(17)</sup>. Lead or lag networks are usually employed to ensure that: the dominant roots are at desired locations; and the steady-state error for prescribed inputs (ramp, step) is less than a given value.

If only the first of the above conditions is to be satisfied, there will be an infinity of solutions. With both conditions, the solution is unique (when the form of the compensation network is prescribed).

For the problem to be treated here, the first of the above conditions is retained, but the second is replaced by a specification concerning the root sensitivity.

This new condition may take different forms (depending on the particular problem), such as sensitivity of the relative damping factor or natural frequency. The method is due to Rung and Thaler<sup>(8)</sup> and is developed in the following.

### 3.2.1.1 The U Circle

We consider the sensitivity of the closed-loop poles in the manner discussed in Sec. 3.1.2. The pertinent definitions, (29) and (30), are repeated here for convenience.

$$S_{z_j}^i = \frac{S_K^i}{(z_j - q_i)} \quad (108)$$

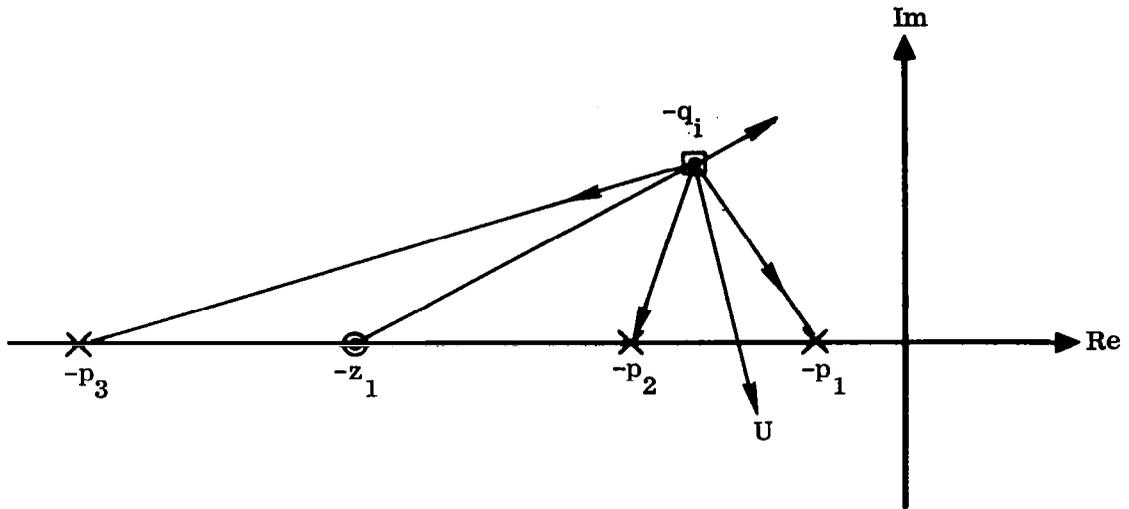
$$S_{p_j}^i = \frac{S_K^i}{(q_i - p_j)} \quad (109)$$

These satisfy relation (36), which may be written as

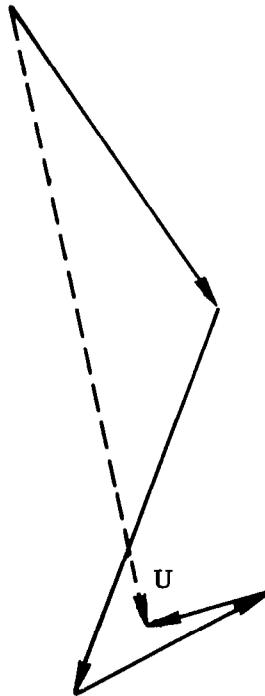
$$\sum_{j=1}^n \frac{1}{(z_j - q_i)} + \sum_{j=1}^{m+n} \frac{1}{(q_i - p_j)} = \frac{1}{S_K^i} \quad (110)$$

The above three relations suggest a simple graphical procedure for determining the root sensitivities. From the closed-loop pole,  $-q_i$ , a vector can be drawn towards each open-loop pole and away from each open-loop zero. The magnitude of each vector must be equal to the inverse of the distance from  $-q_i$  to the pole or zero involved. (See Fig. 11.) By adding all these vectors (see diagram b of Fig. 11), we obtain a vector sum,  $U$ , which by virtue of (110) is precisely the vector  $1/S_K^i$ . It is easy to see, therefore, that the sensitivity of  $-q_i$  to each pole or zero is equal to the vector just drawn to the particular pole (or away from the particular zero) divided by the vector  $U$ . A given configuration of open-loop poles and zeros thus implies a unique vector  $U$  associated with the system.

Consider now a compensation procedure whereby a pole and zero are added to the negative real axis. If the prime purpose is to obtain a prescribed form of response characteristics, the location of the compensating pole and zero is, in general, not unique. However, if we add the stipulation that the closed-loop root sensitivity is to be (in some sense) minimized, then a unique solution is obtained. One type of sensitivity is expressed by the requirement that the  $U$  vector be maximized. Prior to investigating this question in detail, we derive a fundamental property of the  $U$  vector locus when a compensating pole and zero are added.



(a)



NOTE: Scale in diagram (b) is 2-1/2 times that in diagram (a).

(b)

Figure 11. Determination of U Vector

Suppose that  $-q = -c \pm jd$  represents the desired location of the dominant closed-loop roots. Suppose further than this requires a phase shift of  $\varphi$  degrees, which is to be contributed by the compensating pole and zero placed along the negative real axis in the  $s$  plane. If  $\varphi > 0$ , a lead network is required, while if  $\varphi < 0$ , a lag network is required. In this way, the desired root location will lie on the revised root locus.

Fig. 12 shows the pole-zero configuration for an uncompensated system, together with a nominal closed-loop pole (solid square) and the desired dominant pole (point Q). Let  $\varphi$  denote the phase lead that must be contributed by the compensating pole and zero. Vector QI represents the uncompensated U vector; that is, before the compensating pole and zero are added. Then

"As the location of the compensating pole and zero is varied, subject to the restriction that  $\varphi$  degrees of phase shift be contributed at point Q, the locus of the U vector is a circle with center at I and radius  $R = (1/d) \sin \varphi$ . This circle will be called the U circle."

To prove this we focus attention on the sensitivity vectors for the compensating pole and zero. In Fig. 13, these are denoted by QM and VQ respectively. It will be shown first that V and M move a circle with radius  $r = \frac{1}{2d}$ . Applying the law of cosines, we have

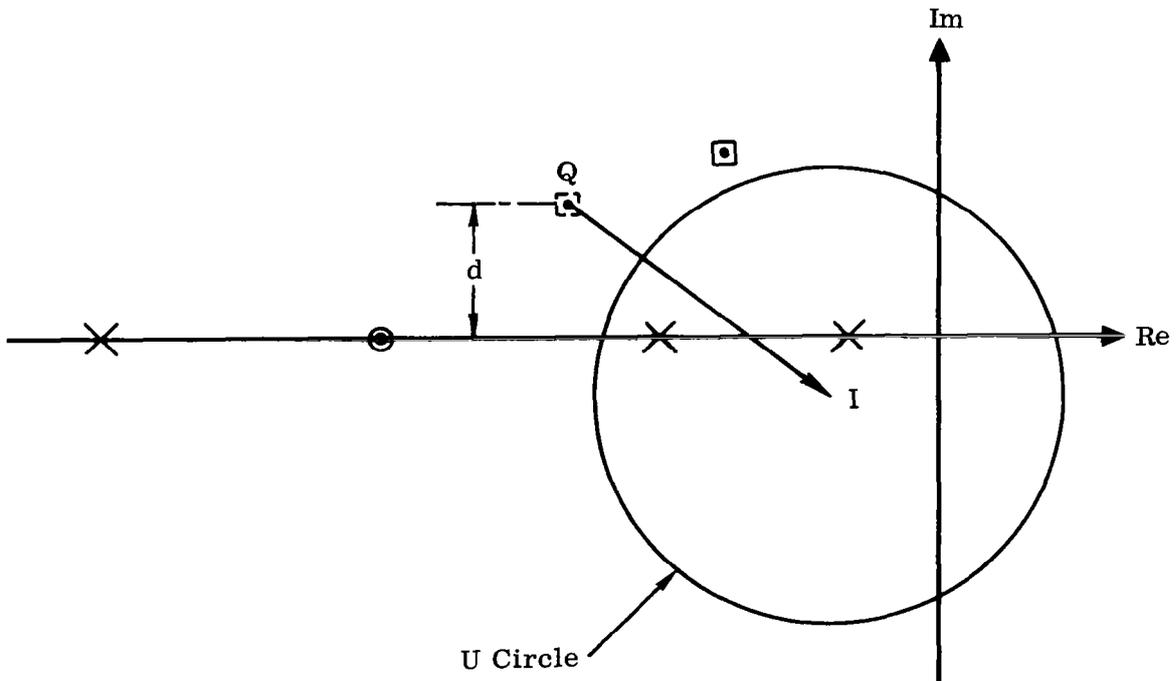


Figure 12. Determination of U Circle



Solving for  $r$ ,

$$r = \frac{1}{2d}$$

which means that we may express  $e$  as

$$e = 2r \sin \varphi \equiv \text{constant}$$

An elementary geometric theorem then shows that  $Q$ ,  $M$ , and  $V$  always lie on a fixed circle (for fixed  $d$  and  $\varphi$ ), which is here referred to as the M circle.

The sum of the pole and zero sensitivity vectors is  $Q\dot{N}$ . This must be added to the uncompensated  $U$  vector,  $QI$ , of Fig. 12 to obtain the  $U$  vector for the compensated system. Since the magnitude of  $Q\dot{N}$  is constant, the tip of this final  $U$  vector describes a circle of radius

$$e = 2r \sin \varphi = \frac{\sin \varphi}{d}$$

Q. E. D.

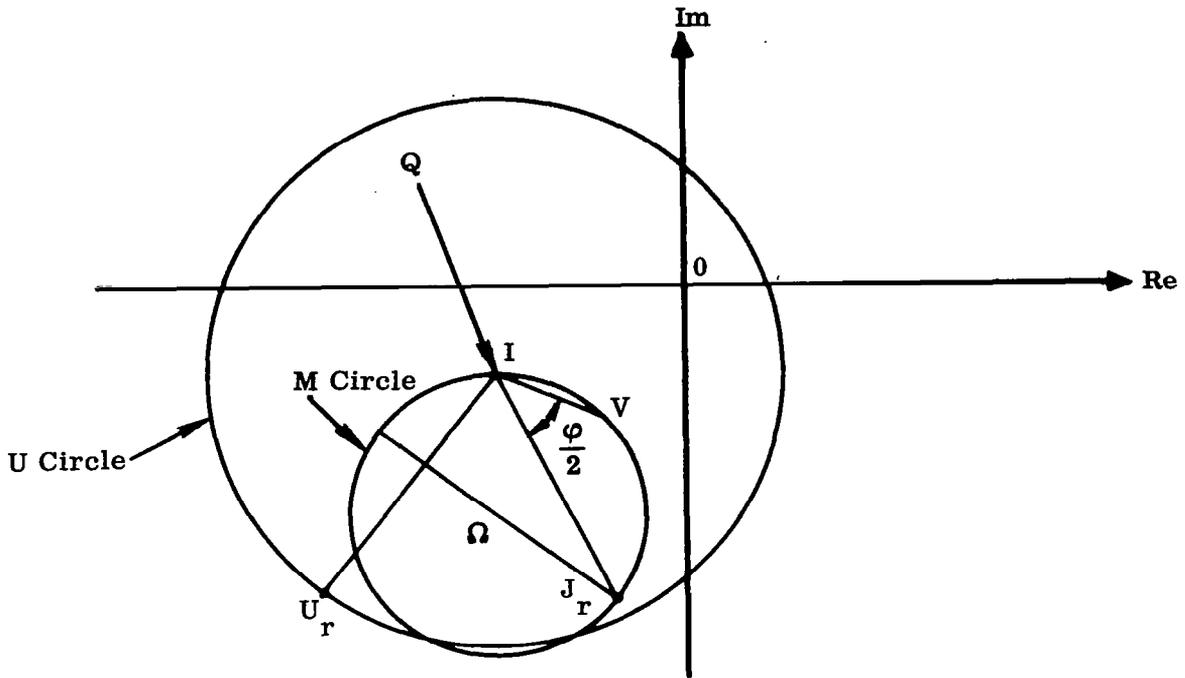
### 3.2.1.2 U Circle Limits

The fact that the compensating pole and zero are restricted to lie on the negative real axis in the  $s$  plane means that the  $U$  vector locus will traverse only a portion of the  $U$  circle. To determine these limits, consider Fig. 14a, in which the compensating zero is located at the origin. To obtain the resulting  $U$  vector, draw  $IV$  parallel to  $QO$ . Now locate point  $J_R$  by constructing  $\angle V I J_R = \varphi/2$ . From  $J_R$ , draw a line that passes through the center,  $\Omega$ , of the  $M$  circle. Finally, draw  $I U_R$  perpendicular to  $J_R \Omega$ . Point  $U_R$  on the  $U$  circle represents the right-hand limit of the  $U$  vector locus.

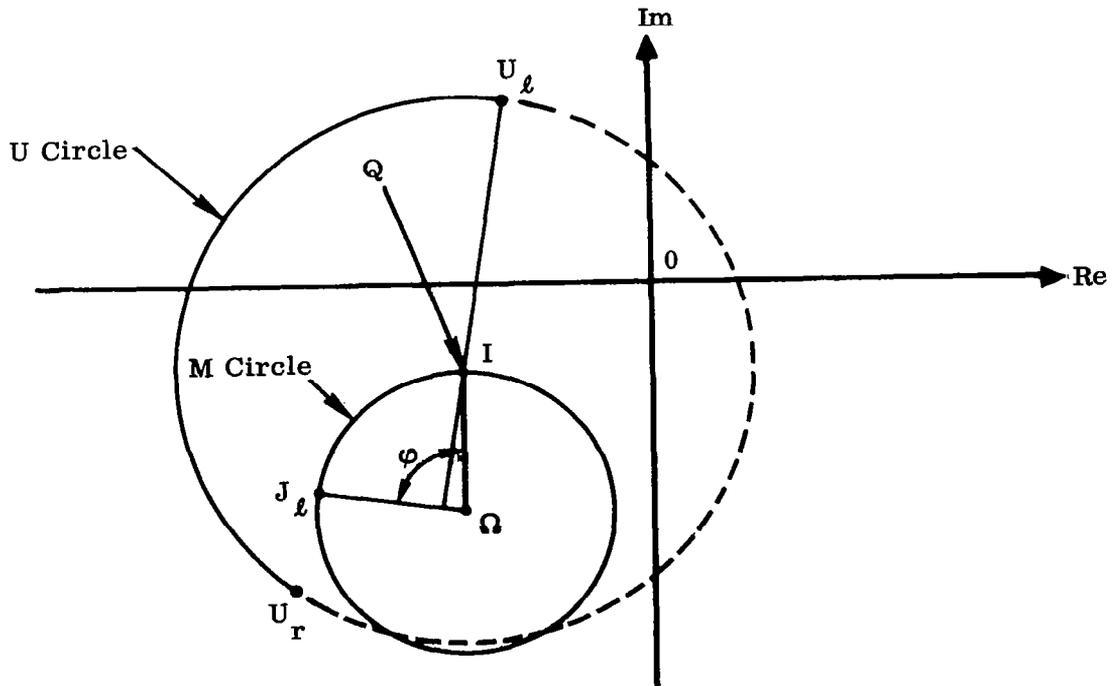
The left-hand limit is obtained from Fig. 14b. Here the compensating pole is located at minus infinity. Locate point  $J_L$  on the  $M$  circle by constructing  $\angle I \Omega J_L = \varphi$ . Draw the line,  $I \Omega$ , that intersects the  $U$  circle at point  $U_L$ . Point  $U_L$  represents the left-hand limit for the  $U$  vector locus.

Thus, as the compensating pole and zero move from the extreme right position (where the compensating zero is at the origin) to the extreme left position (where the compensating pole is at minus infinity), the  $U$  vector follows that portion of the  $U$  circle that is shown solid in Fig. 14b.

These results are easily established by elementary geometry.



(a) Right Hand Limit: Compensating Zero at Origin



(b) Left Hand Limit: Compensating Pole at Minus Infinity

Figure 14. U Circle Limits

### 3.2.1.3 Design Techniques

It will now be shown how the U circle may be used to satisfy stipulated specifications on the sensitivity of the desired root location. The required sensitivity may be expressed in one of several ways:

- a. The sensitivity of the root at Q is to be a minimum with respect to variations in the open-loop poles and zeros. This means that the U vector of the system must be a maximum. In Fig. 14b, for example, the U vector of maximum magnitude is given by  $Q U_r$ , i. e., with a compensating pole at infinity.
- b. The relative damping must be constant when the open-loop gain fluctuates. Since the U vector is tangent to the compensated root locus at point Q, this condition requires that the U vector (extended if necessary) pass through the origin of the s plane.
- c. The relative damping factor must be constant when a particular pole, say  $p_\beta$ , varies. A simple analysis shows that in this case, the U vector must be chosen so that the phase of  $S_{p_\beta}^i$  is equal to the phase of the desired dominant root,  $-q_i$  (or the phase of  $-q_i$  plus  $180^\circ$ ).

Each of the above specifications requires that the compensating pole and zero be found after the U vector is selected. This is merely a converse of the procedures already discussed. Consider, for example, the situation shown in Fig. 15. Suppose that  $Q U_c$  represents the desired U vector, determined in accordance with some prescribed procedure. If the compensation is to introduce a phase lead of  $\varphi$  degrees at Q, then the compensating zero is to the right of the compensating pole. To locate the compensating zero, one proceeds as follows. Through  $\Omega$  (the center of the M circle) draw a line perpendicular to  $I U_c$  and intersecting the M circle at J. Obtain points V and M such that  $\angle M \Omega J = \angle J \Omega V = \varphi$ . Then, through Q, draw a line parallel to IV. The intersection of this line with the real axis locates the compensating zero. The compensating pole is then found by drawing a line through Q and parallel to IM.

Each of the three sensitivity specifications listed above leads to the prescribed U vector, from which the locations of the compensating pole and zero are determined. The general procedure is perhaps best illustrated in terms of a specific example.

Example 3: Given the open-loop transfer function

$$\frac{K}{s(s+1)(s+4)}$$

The system dynamic specifications lead to the requirement that the dominant roots be located at  $-1 \pm j$ . Fig. 16 shows the open-loop pole configuration in the s plane,



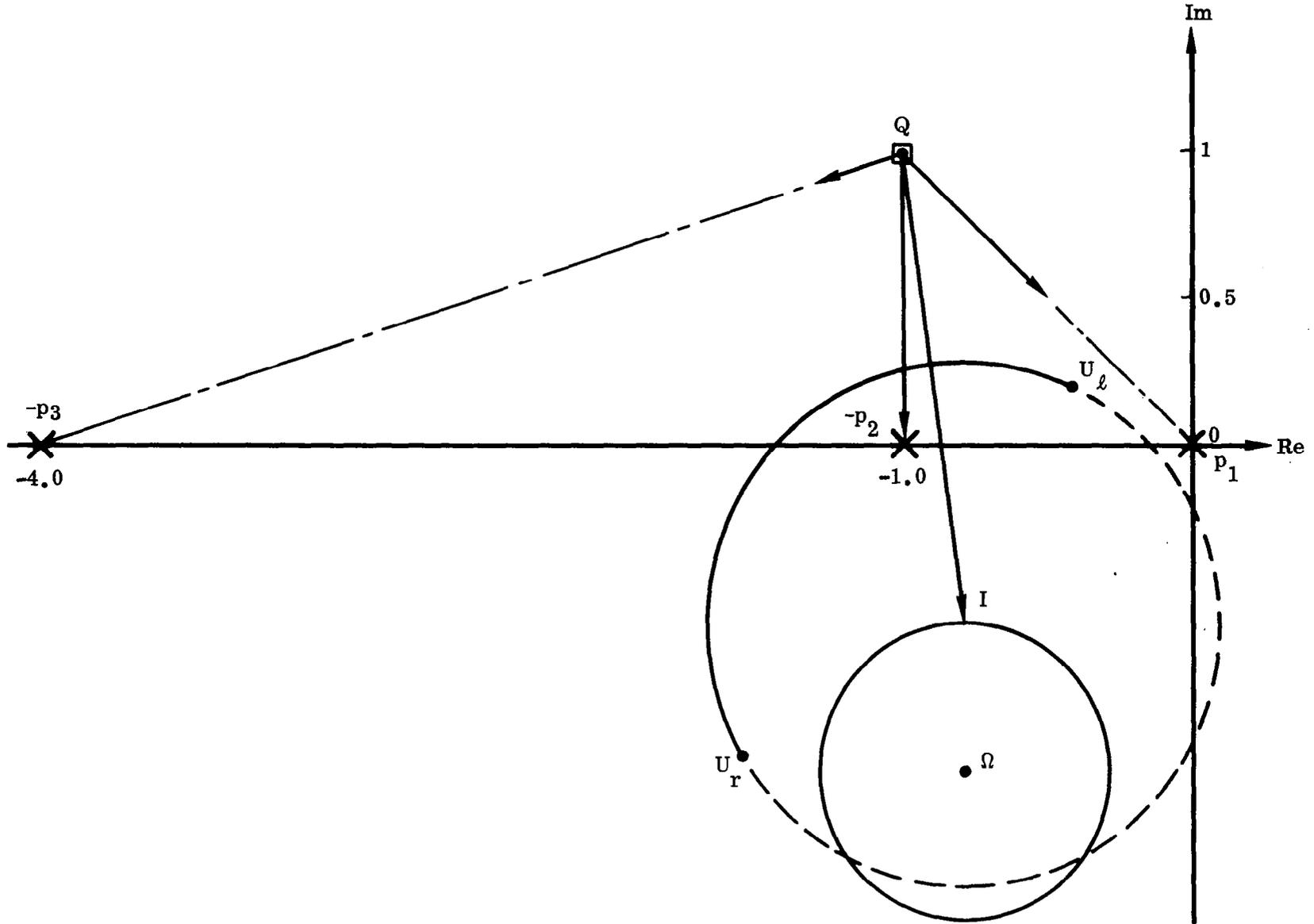


Figure 16. M and U Circles for Example 3

We first take the case where the sensitivity of the root at  $Q$  is to be a minimum with respect to variations in the open-loop poles. As noted earlier, this simply means that the vector from  $Q$  to the  $U$  circle (i. e., the  $U$  vector) must be a maximum. An inspection of Fig. 16 indicates immediately that the maximum  $U$  vector is given by  $\overline{Q\bar{U}_r}$ ; i. e., at the right limit point of the  $U$  circle. Consequently, the compensator for this case has a zero at the origin and a pole at  $-1.30$  (the latter determined by the initial requirement that  $63^\circ$  of phase lead be contributed at point  $Q$ ).

As an alternate sensitivity specification, we may require that the relative damping factor for the dominant root be constant for variations in open-loop gain. Since the  $U$  vector is tangent to the compensated root locus at  $Q$ , this would require that the  $U$  vector be in the direction of line  $\overline{Q0}$  (constant relative damping). However, we note from Fig. 16 that the limiting position of  $\overline{Q\bar{U}}$  is given by  $\overline{Q\bar{U}_l}$  (i. e., the left limit point of the  $U$  circle). This is therefore the best that can be obtained. Since this location requires a pole at infinity, the compensating zero is located on the negative real axis such that  $\varphi = 63^\circ$  of phase lead is contributed at  $Q$ . This corresponds to a zero at  $-1.50$ .

Finally, we consider the requirement that the relative damping factor for the closed-loop pole be constant with respect to variations of pole  $-p_2$ . As previously noted, this implies that the phase of  $S_{p_2}^i$  be equal to the phase of the closed-loop root at  $-q_1$  (denoted in Fig. 16 by point  $Q$ ). From the definition of the root sensitivity, Eq. (109),

$$S_{p_2}^i = \frac{S_K^i}{(q_1 - p_2)} = \frac{1}{\begin{pmatrix} 1 \\ S_K^i \end{pmatrix} (q_1 - p_2)}$$

In Fig. 17, the vector from point  $Q$  to the  $U$  circle represents  $1/S_K^i$ . Vector  $\overline{QA}$  represents  $q_1 - p_2$ . Furthermore, the magnitude and phase angle of the closed-loop pole,  $-q_1$ , is depicted by the vector  $\overline{0Q}$ . We may therefore write

$$S_{p_2}^i = \frac{1}{\overline{QU} \times \overline{QA}}$$

and

$$\angle S_{p_2}^i = -\angle \overline{QU} - \angle \overline{QA} = \angle \overline{0Q}$$

Consequently, the phase angle of  $\overline{QU}$  (the  $U$  vector) is given by

$$\angle \overline{QU} = -\angle \overline{QA} - \angle \overline{0Q} = -270^\circ - 135^\circ = -405^\circ = -45^\circ$$

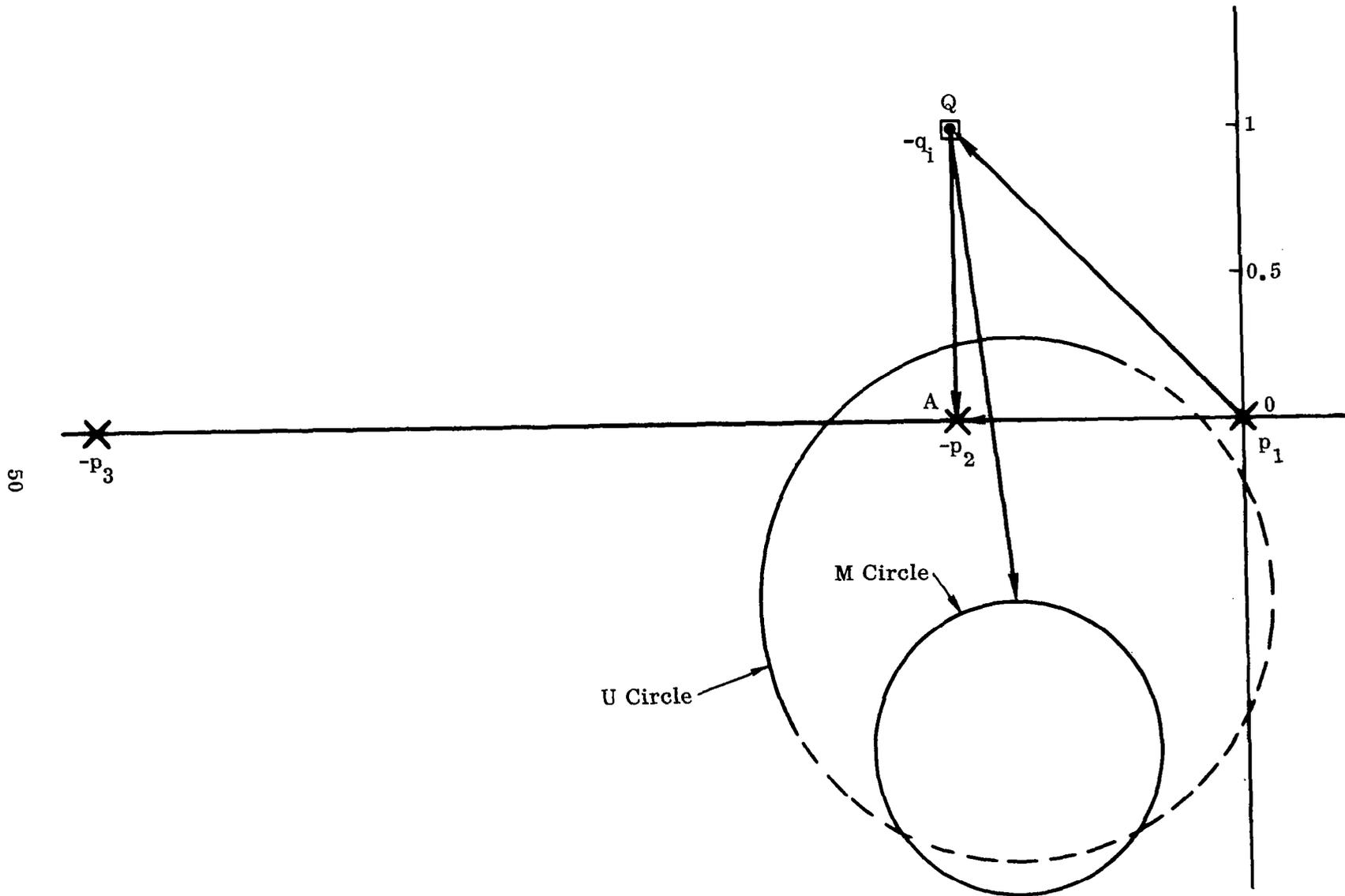


Figure 17. Construction for Constant Relative Damping

In other words, the U vector must be aligned with  $\overline{0Q}$ . This is the same result obtained for constant relative damping factor with respect to open-loop gain variations. In the present case, the result is intuitively plausible, since it implies that the compensating zero is to be placed as close as possible to pole  $p_2$ . This has the effect of minimizing the influence of  $p_2$  on the transient response of the root at Q.

### 3.2.2 Sensitivity in the Large

The discussions thus far have emphasized either implicitly or explicitly that the sensitivity techniques developed are valid only for small excursions of a particular parameter from nominal. This is, in fact, a direct consequence of how sensitivity is defined. The basic definition, Eq. (1), is expressed in terms of differential quantities, from which it is inferred that an equation such as (4) cannot be used for large parameter excursions. It can be shown quite easily, however, that the restriction to small deviations is easily removed.

In the feedback control system of Fig. 18, suppose that the plant transfer function,  $P(s)$ , varies because of fluctuations in the plant parameters. Let<sup>†</sup>

$$\Delta P = P - P_0 \quad (111)$$

$$\Delta T = T - T_0 \quad (112)$$

where the zero subscripts indicate values for nominal plant parameters and T is the closed-loop transfer function

$$T = \frac{G P}{1 + L} \quad (113)$$

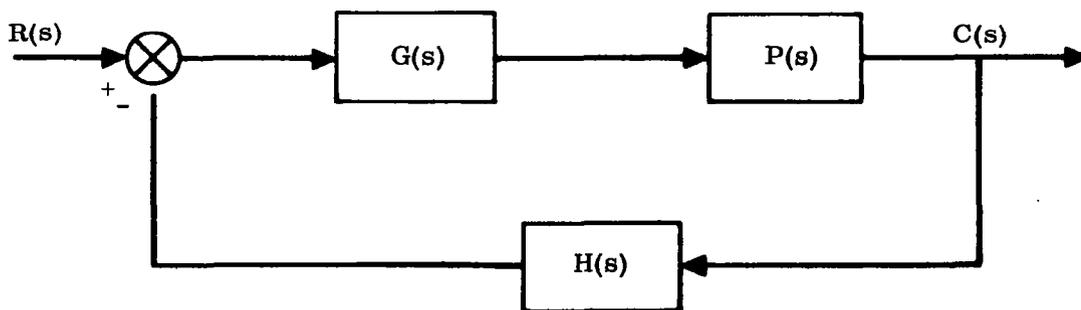


Figure 18. Conventional Feedback Control System

<sup>†</sup> For ease of writing, the argument, s, in the transfer functions will hereafter be dropped.

$$L = GHP \quad (114)$$

The  $\Delta$  quantities are assumed finite and not necessarily small.

Now define the sensitivity of T to variations in P as

$$S_p^T = \frac{\Delta T}{\Delta P} \frac{P}{T} \quad (115)$$

By substituting (111) - (113) in (115), we find, after some reduction,

$$S_p^T = \frac{1}{1 + L_0} \quad (116)$$

Here we have not used the relation

$$\frac{\partial T}{\partial P} = \frac{G}{(1 + GHP)^2}$$

which was used to derive Eq. (4). Therefore, Eq. (116) is valid whatever the magnitude of  $\Delta P$ .

It was first pointed out by Horowitz<sup>(15)</sup>, <sup>(19)</sup> that a failure to appreciate the far-ranging significance of the sensitivity relation for large parameter variations, (116), has led to the formulation of ambiguous and superficial "adaptive" systems that are, in fact, less effective than conventional linear types. He quotes several motivations in the literature to justify the needs for adaptive methods:

"... it is generally taken for granted that the dynamic characteristics of the process will change only slightly under any operating conditions encountered during the lifetime of the control system. Such slight changes are foreseen and are usually counteracted by using feedback. Should the changes become large, the control equipment as originally designed may fail to meet performance specifications<sup>(23)</sup>.

The use of feedback in the classical sense ... may be considered passive adaptation ... For more complex system or systems whose environment is more severe, simple passive adaptation may not be sufficient<sup>(22)</sup>.

If, however, the parameter variations are extremely large, the gain required to maintain the specified system performance may become so high as to be unobtainable because of noise or saturation limits<sup>(24)</sup>.

For systems with more complicated dynamics, it is usually not possible to "swamp out" the fixed elements characteristics and still obtain the desired performance . . . . The use of conditional feedback, sometimes referred to as the "model approach," allows the designer to minimize these effects . . . . Even the best of linear designs break down, however, when the parameters vary over too wide a range<sup>(25)</sup>.

The adaptive approach would aim to maintain a prescribed sensitivity or performance criterion in the face of process changes. . . . The adaptive viewpoint would be especially suited to the design of controllers for processes whose dynamics are not completely known in advance<sup>(26)</sup>.

Conventional control systems are designed to meet certain specifications under certain given conditions of the environment and the system parameters, but should these conditions change the performance will change as a result<sup>(27)</sup>.

To examine these arguments critically, we consider the classical feedback configuration in which  $G(s) \equiv 1$  in Fig. 18.

The sensitivity is

$$S_P^T = \frac{1}{1 + HP_0}$$

while the desired closed-loop transfer function is

$$T_0 = \frac{P_0}{1 + HP_0} = P_0 S_P^T$$

For this configuration it is impossible to realize  $S_P^T$  independently of  $T_0$ . Consequently, one must compromise either the sensitivity or the desired closed-loop properties of the system. This fundamental property of the system is directly related to the fact that there is only one "degree of freedom" for this configuration; i. e., only one free compensating network is available.

When  $G(s) \neq 1$  in Fig. 18, then  $G$  and  $H$  may be chosen so that  $S_P^T$  and  $T$  are realized independently of one another. For example, from (116),

$$L_0 = \frac{1}{S_P^T} - 1 \tag{117}$$

Thus H may be selected to satisfy the sensitivity constraint while G may be chosen to yield the desired closed-loop requirement from

$$T_0 = \frac{G P_0}{1 + L_0} = G P_0 S_P^T \quad (118)$$

Here there are two "degrees of freedom." For this case, "it appears that it is theoretically possible to design an ordinary type of feedback system to have a chosen sensitivity to any amount of plant variation."<sup>(19)</sup> Many different types of two-degree-of-freedom configurations are possible. (15) The essential point is that they are all essentially equivalent in having the capability to realize T and  $S_P^T$  independently. With this point in mind, it becomes fruitless to search for new and exotic configurations (i.e., model feedback, reference model, etc.) when what is really needed is merely an additional degree of freedom.

As is generally the case, added benefits are obtained at a price. In the situation considered above, the desired insensitivity of T to large variations in P is obtained at the cost of large open-loop gain, which effectively means compensating networks with large bandwidth. The design philosophy of the method will be explained in terms of familiar root-locus concepts.

### 3.2.2.1 Design in the s Plane

A typical problem in control system design is usually expressed as the need to maintain a few dominant closed-loop roots relatively invariant while system parameters vary over wide ranges. For definiteness, let us consider the plant transfer function given by

$$P = \frac{K}{s(s^2 + 2\zeta_p \omega_p s + \omega_p^2)} \quad (119)$$

It is assumed that K may vary by a factor of 4, while  $\zeta_p$  and  $\omega_p$  are such that the poles of P may be anywhere in the rectangles ABCD,  $\bar{A}\bar{B}\bar{C}\bar{D}$  of Fig. 19. It is desired that the dominant closed-loop pole pair be located within a circle of radius 1.2, centered at  $-10 \pm j10$  ( $R, \bar{R}$  in Fig. 19).

One way of achieving this is to place a complex zero pair ( $z_1, \bar{z}_1$ ) in the vicinity of ( $R, \bar{R}$ ) and to use a high open-loop gain. There are two fundamental problems:

- a. Where to place the complex zero pair.
- b. Where to place the poles associated with the zeros (since the compensator must be physically realizable).

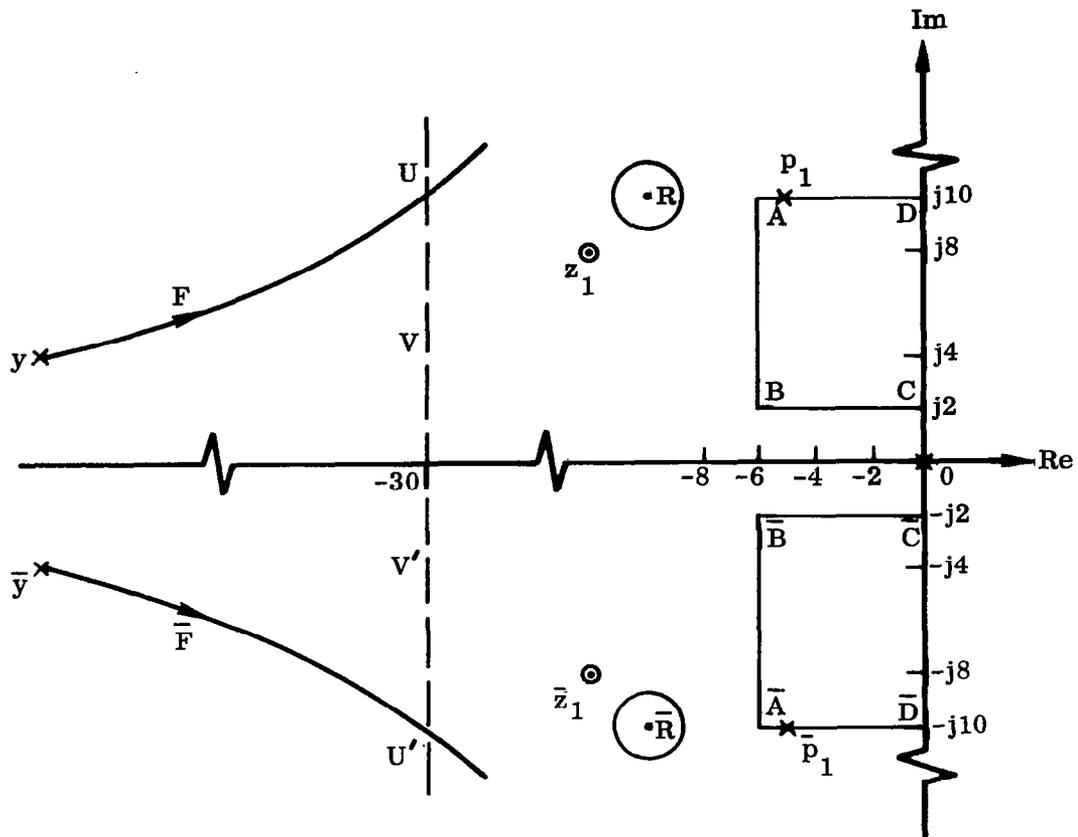


Figure 19. Design Philosophy for Invariance of Dominant Closed-Loop Poles

The use of high open-loop gain means that ultimately the root loci associated with the compensator poles are such that a closed loop may be in the vicinity of the dominant poles or else in the right-half plane (unstable) if the gain is sufficiently high. Consequently, these compensator poles must be "sufficiently far" to the left. The further they are to the left (and the higher the gain), the greater the bandwidth of the system. Usually, one stipulates that the compensator closed-loop poles should be to the left of some arbitrary line,  $UVV'U'$ , in the  $s$  plane. There is some freedom in choosing this line, depending on the particular application; however, this choice is an integral part of the design procedure, and once made, serves as a primary constraint to be satisfied.

It is required therefore to:

- a. Choose  $(z_1, \bar{z}_1)$  such that the desired  $T$  is obtained; and
- b. Choose the compensator pole pair  $(y, \bar{y})$  such that the desired sensitivity is achieved.

A simple analysis shows that a two-degree-of-freedom structure is required. That is, if the compensator poles and zeros were contained within G or H alone (with the other identically one), then  $z_1$  and  $y$  could not be manipulated to control both sensitivity and desired root location independently.

Furthermore, of the infinity of possible compensator pole locations, we seek to select the one that results in minimum bandwidth. The design thus proceeds in two stages:

- a. Locate the compensator zero pair ( $z_1, \bar{z}_1$ ).
- b. Locate the compensator pole pair ( $y, \bar{y}$ ).

We consider these in turn.

### 3.2.2.2 Compensator Zero Location

The location of the compensator zeros is influenced by the variation in open-loop gain,  $K$ , and by the drift of the open-loop poles of  $P$ . Consider first the variation in  $K$ . Suppose that  $E$  is a root of  $1 + L = 0$  when  $K$  takes a particular value. Then (see Fig. 20)

$$K = \frac{\prod_i (p_i E)}{\prod_j (z_j E)}$$

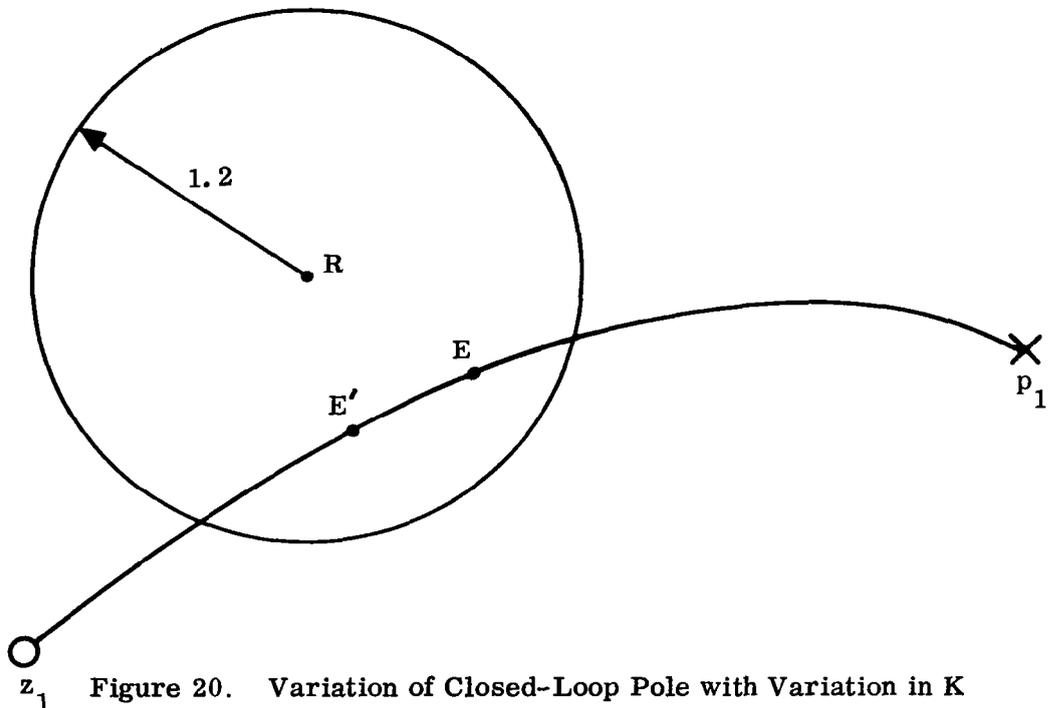


Figure 20. Variation of Closed-Loop Pole with Variation in  $K$

which is merely the usual expression for open-loop gain as the product of pole vectors to the closed-loop pole divided by the product of zero vectors to the same closed-loop pole. When  $K$  is increased to  $K'$ , the closed-loop pole moves to point  $E'$ , and we have

$$K' = \frac{\prod (p_i E')}{\prod (z_j E')}$$

The conditions of the problem are such that the drift in  $E$  is very small; therefore  $p_i E \approx p_i E'$  and  $z_j E \approx z_j E'$  except for the zero,  $z_1$ , near  $E$ . Consequently,

$$\frac{K'}{K} \approx \frac{(z_1 E)}{(z_1 E')} \quad (120)$$

To find the change in the dominant root of  $1 + L = 0$  due to change in the position of the complex pole of  $P$ , let  $p_1$ ,  $\bar{p}_1$  and  $p_1'$ ,  $\bar{p}_1'$  respectively denote the old and new complex pole positions of  $L$ . Also, let  $E$ ,  $E'$  denote the old and new positions of the dominant root of  $1 + L = 0$ . Then

$$\begin{aligned} K &= \frac{(p_1 E)(\bar{p}_1 E) \prod_{i \neq 1} (p_i E)}{(z_1 E) \prod_{j \neq 1} (z_j E)} = \frac{(p_1' E')(\bar{p}_1' E') \prod_{i \neq 1} (p_i E')}{(z_1 E') \prod_{j \neq 1} (z_j E')} \\ &\approx \frac{(p_1' E')(\bar{p}_1' E') \prod_{i \neq 1} (p_i E)}{(z_1 E') \prod_{j \neq 1} (z_j E)} \end{aligned}$$

Therefore

$$\frac{(z_1 E')}{(z_1 E)} \approx \frac{(p_1' E')(\bar{p}_1' E')}{(p_1 E)(\bar{p}_1 E)}$$

This approximation is valid whenever  $|p_i E| \approx |p_i E'|$  except possibly for  $i = 1$ , and when  $|z_j E| \approx |z_j E'|$  except for  $j = 1$ . As noted earlier, the design constrains the drift in  $E$  to be small, which means that these conditions are generally satisfied. In this case, it is also permissible to assume that  $p_1' E' \approx p_1' E$  and  $\bar{p}_1' E' \approx \bar{p}_1' E$ . We then have

$$\frac{(z_1 E')}{(z_1 E)} = \frac{(p_1' E)(\bar{p}_1' E)}{(p_1 E)(\bar{p}_1 E)} \quad (121)$$

Equations (120) and (121) are used to find the approximate shape and orientation of the region of variation of the dominant roots of  $1 + L = 0$  due to variations in gain and the poles of  $L$ . This is done as follows.

In Fig. 21, point  $O$  corresponds to the location of the compensating zero, and  $X$  corresponds to the nominal location of the dominant root. These two points may be selected at will, since only their relative location determines the scale and orientation of Fig. 21. It is assumed in the following that  $K_{\min}$  represents the nominal value of  $K$ . As drawn in Fig. 21,  $\overline{OX} = 1.0 \angle 0^\circ$ .

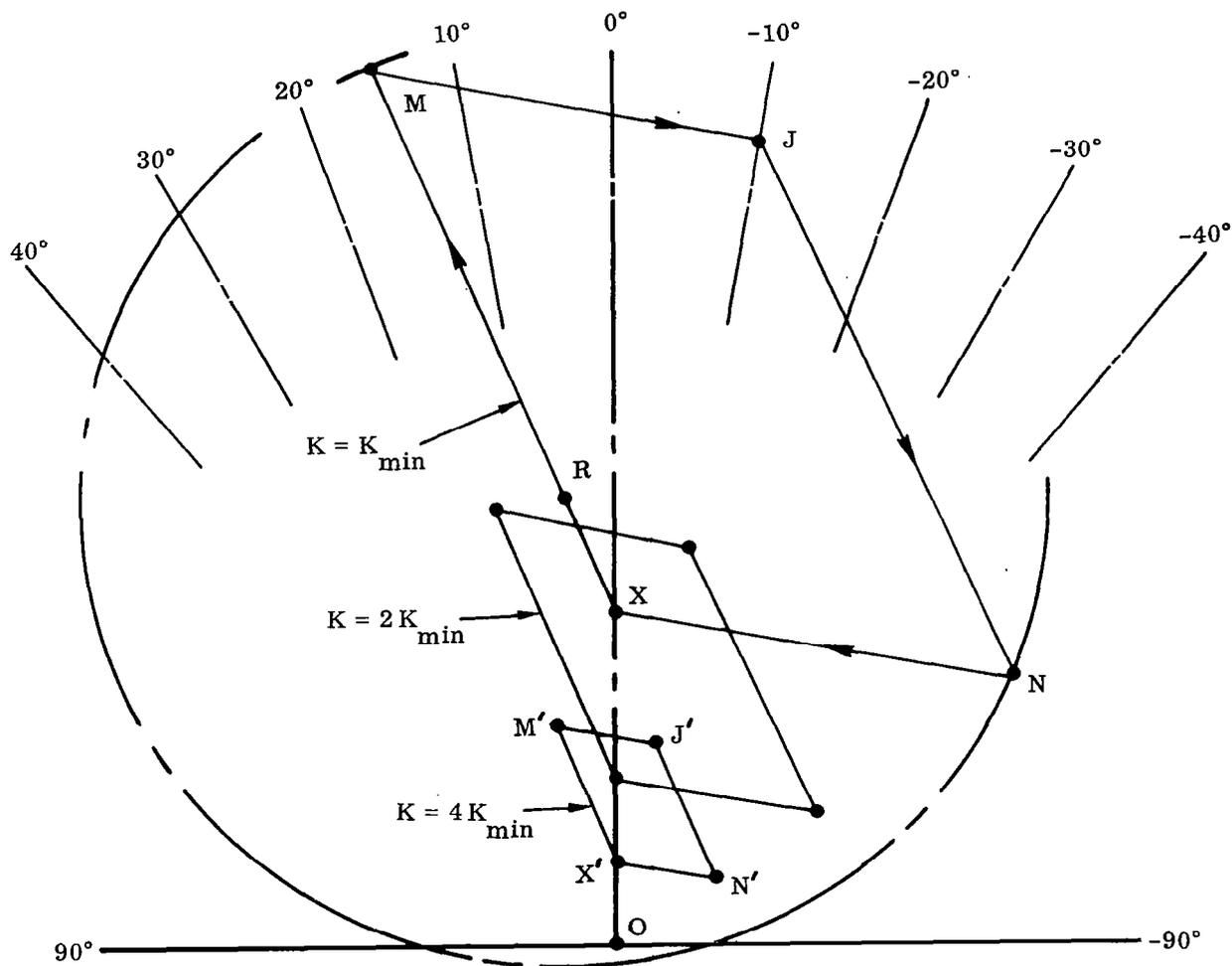


Figure 21. Region of Variation of Dominant Roots

Now suppose that  $K$  changes from its nominal value to  $K' = 4K_{\min}$ . The new position of the dominant root (denoted by  $X'$ ) is given by Eq. (120); viz.,

$$\frac{K'}{K} \approx \frac{z_1 X}{z_1 X'} = \frac{\overline{OX}}{\overline{OX}'} = 4$$

This serves to locate  $X'$  in Fig. 21, i. e.,  $\overline{OX}' = 0.25 \overline{OX}$ . Thus  $X'X$  is the approximate locus of the dominant root as  $K$  changes from  $K_{\min}$  to  $4K_{\min}$ .

With the open-loop gain equal to its nominal value,  $K_{\min}$ , we now seek to determine the locus of the dominant root as the open-loop pole traverses rectangle  $ADBC$  of Fig. 19. Consider first the effect of the plant poles moving from  $A$  to  $D$ . We use Eq. (121)† with (see Fig. 19)

$$p_1 E \approx -4$$

$$\bar{p}_1 E \approx -4 + j20$$

$$p_1' E \approx -10$$

$$\bar{p}_1' E \approx -10 + j20$$

In these relations,  $E$  (and  $E'$ ) is assumed located at  $R$  in Fig. 19 (which also accounts for the "approximately equal to" symbol being used). Therefore

$$\frac{(z_1 E')}{(z_1 E)} = \frac{\overline{OM}}{\overline{OX}} = \frac{(-10)(-10 + j20)}{(-4)(-4 + j20)}$$

and

$$\overline{OM} = 2.70 \angle 15^\circ$$

since  $\overline{OX} = 1.0 \angle 0^\circ$  by definition. This locates point  $M$  in Fig. 21. With  $K = 4K_{\min}$  and the plant pole still at  $D$ , Eq. (120) is used to determine  $M'$ ; viz.,

$$\frac{z_1 E}{z_1 E'} = \frac{\overline{OM}}{\overline{OM}'} \approx 0.25$$

† Keeping in mind that  $E(E')$  in Fig. 20 or Eq. (121) corresponds to  $X(X')$  of Fig. 21.

In other words  $\overline{MM'}$  is the approximate locus of the dominant root as  $K$  varies from  $K_{\min}$  to  $4K_{\min}$  when the plant pole is at  $D$ .

Continuing in this fashion, we obtain the diagram of Fig. 21. The quadrilateral,  $XMJN$ , represents the locus of the dominant root when the plant pole traverses rectangle  $ADBC$  in the  $s$  plane (Fig. 19) and  $K = K_{\min}$ . The two other quadrilaterals represent the dominant root loci for the other values of gain shown.

The general shape of the region of variation of the dominant root is now available and is used to determine the required scale of Fig. 21 and its orientation in the  $s$  plane. The dominant poles of  $T$  (roots of  $1 + L = 0$ ) lie inside region  $X'M'JNN'X'$ . By trial and error, the circle of minimum radius that contains this region is found (center  $R$  and radius  $RM$  in Fig. 21). The specifications dictate that the dominant roots of  $1 + L = 0$  must lie inside a circle of radius 1.2 centered at  $-10 + j10$ . The magnitude of  $RM$  is therefore 1.2, and  $R$  must correspond to point  $-10 + j10$ . It is found that  $OR$ , which is the distance of  $z_1$  from  $-10 + j10$ , is also 1.2. It is clear from Fig. 21, that point  $R$  is a root of  $1 + L = 0$  when  $K = K_{\min}$  and when the plant poles are at approximately  $-5 \pm 10j$ . The latter is obtained by noting that in Fig. 21,  $R$  lies on line  $XM$ , which corresponds to line  $AD$  in Fig. 19. By measuring on Fig. 21, it is found that  $XR \approx (0.18)XM$ . Since  $(0.18)(AD) \approx 1$ , point  $-5 + j10$  is thus determined.

With the above information, the required location of the zeros,  $z_1, \bar{z}_1$ , of Fig. 19 can be obtained fairly closely. It is known that the net angle of the vectors from the poles and zeros of  $L$  to  $-10 + j10$  must be  $180^\circ$ . Since the far-off poles of  $L$  are not known as yet, a few degrees may be assigned for their contribution, or, as a first approximation, they may be neglected altogether. Therefore (see Fig. 19):  $180^\circ = \angle OR + \angle pR + \angle \bar{p}R - \angle z_1R - \angle \bar{z}_1R$ . Since  $\angle \bar{z}_1R \approx 90^\circ$ , this leads to  $\angle z_1R \approx 146^\circ$ . It has previously been determined that  $|z_1R| = 1.2$ . Since  $R$  is at  $-10 + j10$ , this locates  $z_1$  at  $-9 + j9.3$ . With this information, it is possible to turn to the problem of locating the far-off poles of  $L$ .

### 3.2.2.3 Location of Far-off Poles

The design specifications require that the far-off poles be located sufficiently to the left of line  $UVV'U'$  (Fig. 19) that no closed-loop pole is to the right of the line for maximum open-loop gain. Suppose that for  $K = 4K_{\min}$ , a closed-loop pole pair is located at  $(W, \bar{W})$  in Fig. 22; i. e., at  $-30 \pm j180$ . At  $W$ , the totality of poles and zeros near the origin appears as a single pole at the origin. The phase contribution of this pole at the origin to the point  $W$  is  $100^\circ$ . Consequently, the two far-off poles must contribute a total of  $180 - 100 = 80^\circ$  to  $W$ . It is easy to see that the locus of pole positions that contribute  $80^\circ$  to  $W$  is a segment of a circle passing through  $W$ . Three points on this circle are quickly located. One must be a point on the negative real axis such that a double pole placed here contributes  $2 \times 40 = 80^\circ$  to  $W$ . (See Fig. 22.) The other two points are on horizontal lines through  $W$  and  $\bar{W}$  such that the first contributes  $0^\circ$  and the second contributes  $80^\circ$  ( $N$  and  $\bar{N}$  in Fig. 22). This enables one to draw the circle shown.

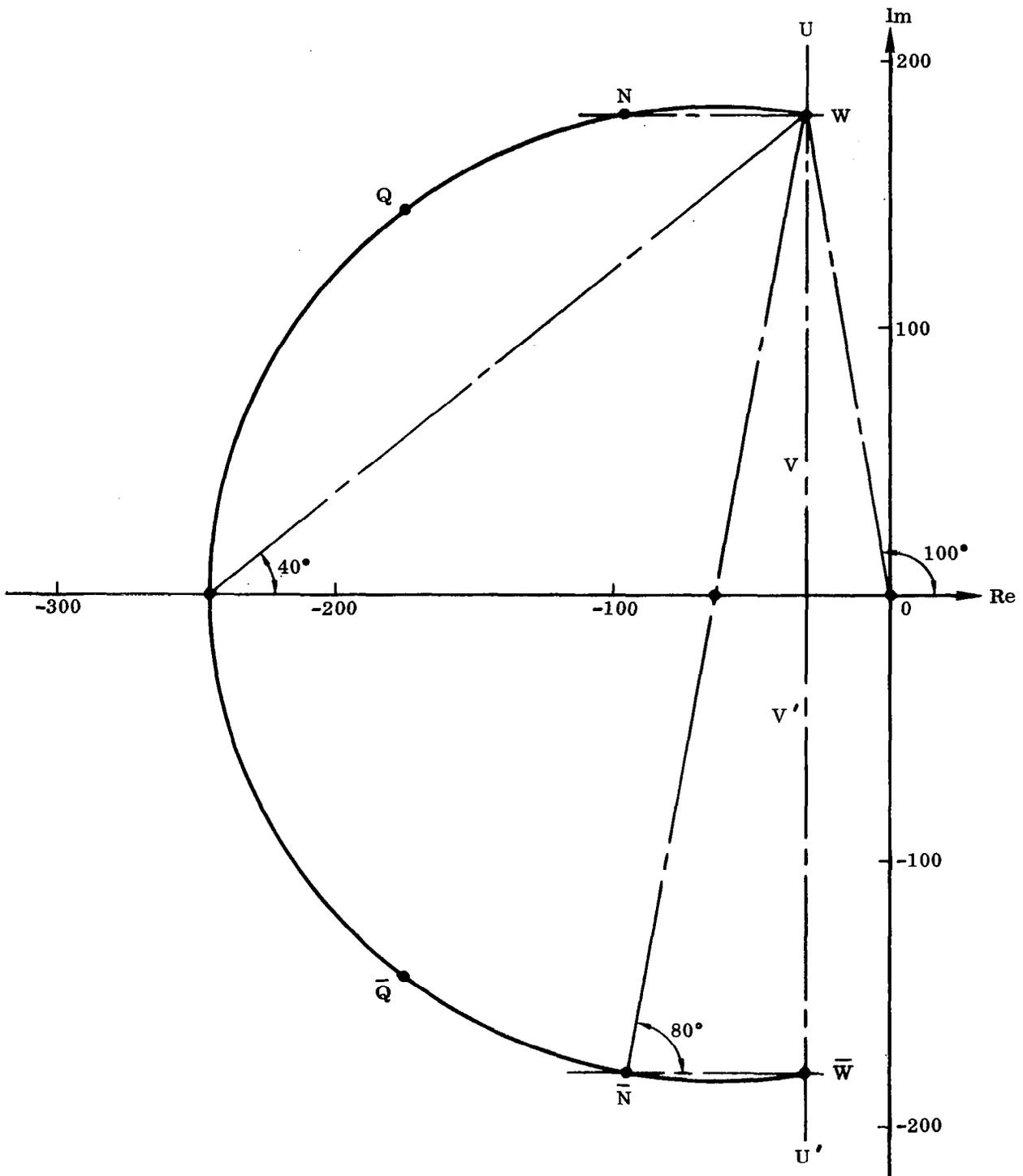


Figure 22. Location of Far-off Poles

Suppose now that the two far-off poles are located at  $(Q, \bar{Q})$ . Since a closed-loop pole is to appear at  $W$  for  $K = 4K_{\min}$ , we must have

$$4K_{\min} = |(QW)(\bar{Q}W)(OW)| \quad (122)$$

However, for a closed-loop pole to appear at  $R$  (Fig. 19), we must have, in addition,

$$K_{\min} = \left| \frac{(p_1 R)(\bar{p}_1 R)(OR)(QR)(\bar{Q}R)}{(z_1 R)(\bar{z}_1 R)} \right| \quad (123)$$

If the  $K_{\min}$  calculated by (122) is substantially equal to the  $K_{\min}$  calculated by (123), then  $(Q, \bar{Q})$  is a permissible location for the poles. If not, another pole location along the circle is selected, and the process is repeated until the two  $K_{\min}$  are in good agreement.

The  $K_{\min}$  thus determined is not necessarily the lowest possible, and these far-off pole locations are therefore not the best in the sense of smallest bandwidth. A new  $W$  point along  $UVV'U'$  is chosen, and the process is repeated until the smallest  $K_{\min}$  (and therefore the minimum bandwidth) is found.

This completes the design procedure.

Remark: The concept of sensitivity in the large appears to shed new light on the question of exotic adaptive techniques for plants that exhibit wide parameter variations. The methods discussed above have been applied to aircraft and to re-entry vehicles<sup>(21), (20)</sup> and have pointedly challenged the capability of adaptive methods to yield a superior system.

As previously mentioned, the price paid for producing a prescribed insensitivity is the large bandwidth of the resulting system. In some cases, an enormous gain-bandwidth requirement may be substantially reduced by first employing minor feedback loops around the plant.

There are possible limitations in the above approach due to noise and signal saturation. In this respect, it should be noted that the amount of plant saturation caused by the useful signal input is determined by the desired system performance and is completely independent of how the performance is to be obtained, whether by ordinary feedback or by means of an adaptive system. The signal level in the plant is determined by the desired output; therefore, any two systems with the same plant, the same desired system transfer function, and the same output must have exactly the same plant-saturation problems with respect to the useful signals, or to the noise that enters at the same point as the useful signal.

Another serious limitation of ordinary feedback (and one that has not received much attention in the adaptive literature) is the inability to realize, even theoretically, any desired loop gain-bandwidth when the plant is nonminimum-phase (actually if it has at least two zeros or poles in the right-half plane). Therefore, it may be impossible to obtain the desired benefits of feedback by means of ordinary feedback structures. In this regard, Horowitz states<sup>(19)</sup> "It would be a genuine and important contribution if it could be shown how these desired benefits (insensitivity, disturbance rejection, etc.), may be obtained by adaptive systems. However, once again, while the above has been rather vaguely cited in the literature as a justification for departing from ordinary feedback, there has not been any corresponding demonstration that the adaptive systems can do any better. One searches in vain in the adaptive literature for a clear-cut, quantitative statement of a problem which is shown to be intractable by ordinary feedback, but amenable to an adaptive design."

### 3.3 SENSITIVITY AND OPTIMAL CONTROL

A general formulation of the optimal control problem is the following. Given a system described by the vector differential equation

$$\dot{x} = f(t, x, u, a) \quad (124)$$

$$x(t_0) = c \quad (125)$$

where  $x$  is a state vector,  $u$  is the control vector, and  $a$  is a vector representing a set of  $m$  plant parameters. The nominal value of the  $i^{\text{th}}$  component of  $a$  will be denoted by  $a_{i0}$ . The problem is to select the control vector,  $u$ , such that the index of performance

$$J(t_0, x, u, a_0) = \int_{t_0}^{t_f} L(t, x, u, a_0) dt + G[t_f, x(t_f)] \quad (126)$$

is a minimum (or maximum). Note that the index of performance is based on the nominal value of  $a$ . Suppose that by one of the usual optimization methods, the optimal control is found to be†

$$u^*(t) = \Psi[t, x(t), a_0] \quad (127)$$

---

† The optimal control shown is closed-loop; that is, it depends on the current state of the system. For open-loop control,  $c$  would replace  $x(t)$  in (127).

We denote the corresponding index of performance by

$$J^*(t_0, x, a_0) = \underset{u}{\text{Min}} J(t_0, x, u, a_0) \quad (128)$$

If the actual parameter vector instead of the nominal  $a_0$  is used, the increment in performance index is given by

$$\Delta J = J^*(t_0, x, a) - J^*(t_0, x, a_0) \quad (129)$$

$\Delta J$  may be positive or negative, depending on how the system parameters vary. In other words, there may indeed be some combination of system parameters that yields an improved index of performance since the latter was optimized with respect to  $u$  only. Expanding  $J^*(t_0, x, a)$  in a Taylor series about the nominal system parameters and discarding all but first-order terms reduces Eq. (129) to

$$\Delta J = \sum_i \left. \frac{\partial J^*(t_0, x, a)}{\partial a_i} \right|_{a_i = a_{i0}} \delta a_i \quad (130)$$

Following Pagurek<sup>(12)</sup>, we define the performance index sensitivity function as

$$S_i^J = \left. \frac{\partial J^*(t_0, x, a)}{\partial a_i} \right|_{a_i = a_{i0}} \quad (131)$$

In certain cases, this sensitivity function may be expressed directly in terms of the given parameters of the system. One such case is the optimal control problem for the linear system

$$\dot{x} = A_0 x + B_0 u \quad (132)$$

$$y = G_0 x \quad (133)$$

$$x(t_0) = c$$

with a quadratic performance index

$$J(t_0, x, u, a_0) = \int_{t_0}^{t_f} (y^T Q y + u^T R u) dt + x^T(t_f) M x(t_f) \quad (134)$$

Here,  $A, B, G, Q, M,$  and  $R$  are constant matrices,  $\dagger$  and the zero subscripts denote nominal values of the parameter vector,  $a$ .

It is known<sup>(30)</sup> that for this problem, the optimal control vector is given by

$$u^*(t) = -R^{-1} B_0^T P_0(t) x(t) \quad (135)$$

where  $P_0(t)$  satisfies the matrix Riccati equation

$$\dot{P}_0 + P_0 A_0 + A_0^T P_0 - P_0 B_0 R^{-1} B_0^T P_0 + G_0^T Q G_0 = 0 \quad (136)$$

$$P_0(t_f) = M \quad (137)$$

and the optimal performance index is

$$J^*(t, x, a_0) = x^T P_0 x \quad (138)$$

These results are also derived in another monograph of the present series<sup>(31)</sup>.

Now when the control, (135), is used with the actual (or perturbed) system parameter vector,  $a$ , instead of the nominal,  $a_0$ , the motion is described by

$$\dot{x} = A x - B R^{-1} B_0^T P_0 x = F(t) x \quad (139)$$

where

$$F(t) = A - B R^{-1} B_0^T P_0 \quad (140)$$

The solution of Eq. (139) is

$$x(t) = \Phi(t, t_0) c \quad (141)$$

where  $\Phi(t, t_0)$  is the transition matrix<sup>(10)</sup> for the system (139). Substituting (135) and (141) in (126), we obtain

$$J^*(t_0, x, a) = c^T \left[ \int_{t_0}^{t_f} \Phi^T(t, t_0) Q_1 \Phi(t, t_0) dt + \Phi^T(t_f, t_0) M \Phi(t_f, t_0) \right] c \quad (142)$$

---

$\dagger$  We also make the usual assumption that the weighting matrices,  $Q, R,$  and  $M,$  are symmetric, positive definite. Actually it is sufficient for  $Q$  and  $M$  to be positive semi-definite.

where

$$Q_1 = G^T Q G + P_0 B_0 R^{-1} B_0^T P_0 \quad (143)$$

Eq. (142) is of the form

$$J^*(t, x, a) = x^T P x \quad (144)$$

where

$$P(t) = \int_t^{t_f} \Phi^T(\tau, t) Q_1(\tau) \Phi(\tau, t) d\tau + \Phi^T(t_f, t) M \Phi(t_f, t) \quad (145)$$

Noting that  $\Phi(\tau, \tau) = I$  (the unit matrix) for all  $\tau$ , we find the boundary condition

$$P(t_f) = M \quad (146)$$

If we differentiate Eq. (145) with respect to  $t$  using Leibnitz's rule, we have

$$\begin{aligned} \dot{P} = & \int_t^{t_f} \frac{d\Phi^T(\tau, t)}{dt} Q_1(\tau) \Phi(\tau, t) d\tau + \frac{d\Phi^T(t_f, t)}{dt} M \Phi(t_f, t) \\ & + \int_t^{t_f} \Phi^T(\tau, t) Q_1(\tau) \frac{d\Phi(\tau, t)}{dt} d\tau + \Phi^T(t_f, t) M \frac{d\Phi(t_f, t)}{dt} - Q_1(t) \end{aligned} \quad (147)$$

Noting that the transition matrix satisfies the relations<sup>(10)</sup>

$$\frac{d\Phi^T(\tau, t)}{dt} = -F^T(t) \Phi^T(\tau, t) \quad (148)$$

$$\frac{d\Phi(\tau, t)}{dt} = -\Phi(\tau, t) F(t) \quad (149)$$

we obtain the following matrix equation for  $P$ .

$$\dot{P} + F^T P + P F + Q_1 = 0 \quad (150)$$

$$P(t_f) = M \quad (151)$$

It is easy to see that if we put  $a = a_0$  (i.e.,  $A = A_0$ ,  $B = B_0$ , etc.), then

$$P(t) \Big|_{a=a_0} = P_0(t) \quad (152)$$

By virtue of (144), the sensitivity function, (131), becomes

$$S_i^J = c^T \frac{\partial P}{\partial a_i} \Big|_{a_i=a_{i0}} c \quad (153)$$

But from (150),

$$\frac{\partial}{\partial a_i} \left( \frac{dP}{dt} \right) + F^T \frac{\partial P}{\partial a_i} + \frac{\partial F^T}{\partial a_i} P + P \frac{\partial F}{\partial a_i} + \frac{\partial P}{\partial a_i} F + \frac{\partial Q_1}{\partial a_i} = 0 \quad (154)$$

We now define

$$P_i(t) = \frac{\partial P}{\partial a_i} \Big|_{a_i=a_{i0}} \quad (155)$$

and assume that

$$\frac{\partial}{\partial a_i} \left( \frac{dP}{dt} \right) = \frac{d}{dt} \left( \frac{\partial P}{\partial a_i} \right)$$

an operation that is valid under mild restrictions on  $P$ .

Then the performance index sensitivity function is given by

$$S_i^J = c^T P_i c \quad (156)$$

where [after putting  $a = a_0$  in Eq. (154) and making use of Eq. (152)]

$$\frac{d}{dt} P_i + F^T P_i + P_i F + Q_2 = 0 \quad (157)$$

$$P_i(t_f) = 0 \quad (158)$$

and

$$Q_2 = \left[ \frac{\partial F^T}{\partial a_i} P_0 + P_0 \frac{\partial F}{\partial a_i} + \frac{\partial Q_1}{\partial a_i} \right]_{a_i = a_{i0}} \quad (159)$$

Thus, having the sensitivity function, (156), the increment in the performance function due to prescribed increments  $\delta a_i$  in the system parameter is obtained from Eq. (130) as

$$\Delta J = \sum_i S_i^J \delta a_i \quad (160)$$

Remark: For the situation analyzed above, the calculation of the performance index sensitivity function is conceptually simple. However, it is readily apparent that in order to obtain numerical results, the use of a computer is virtually mandatory. The theory may also be generalized to include nonlinear systems. In this case, it can be shown<sup>(9)</sup> that quantities  $J^*(t, x, a)$  and  $J^*(t, x, a_0)$  of Eq. (129) each satisfies a type of Hamilton Jacobi equation. However, as may be expected, the computational aspects become overwhelming.

We may note also that the sensitivity function defined in this section is not the only one that could be used. Rohrer and Sobral<sup>(28)</sup> suggest the use of a relative sensitivity function. This is defined as follows.

At the plant parameter,  $a$ , the relative sensitivity for the control,  $u(t)$ , is defined as the difference between the actual value of the performance index and the value that would be obtained if the control were the optimal for the plant parameters,  $a$  (divided by the optimal performance index for normalization).

$$S_a^J = \frac{J(a, u) - J(a, u^*)}{|J(a, u^*)|} \quad (161)$$

Among the obvious advantages of this definition is that  $S_a^J$  is always a positive number. Moreover, the relative sensitivity reduces to zero at the value of the plant parameters for which the control,  $u(t)$ , is optimal. System performance is always compared with an attainable value. This eliminates a disconcerting element in the Pagurek theory above; namely, that off-nominal values of the system parameters may actually improve the performance index. Further studies are necessary to establish the superiority of one method over the other.

#### 4. REFERENCES

1. Truxal, J. G. Automatic Feedback Control System Synthesis, McGraw-Hill Book Co., Inc., New York, 1955.
2. Bode, H. W. Network Analysis and Feedback Amplifier Design, Van Nostrand Co., Inc., New York, 1945.
3. Ur, H. "Root Locus Properties and Sensitivity Relations in Control Systems," IRE Trans. on Automatic Control, Jan. 1960, p. 57-65.
4. Huang, R. V. "The Sensitivity of the Poles of Linear Closed Loop Systems," AIEE Trans., part II, Appl. & Ind., Vol. 77, May 1958, p. 182-187.
5. Mazer, W. M. "Specification of the Linear Feedback Sensitivity Function," IRE Trans. on Automatic Control, June 1960, p. 85-93.
6. McRuer, D. T., and Stapleford, R. L. Sensitivity and Modal Response for Single Loop and Multiloop Systems, Technical Report ASD-TDR-62-812, Air Force Systems Command, Wright-Patterson Air Force Base, Ohio, Jan. 1963.
7. Horowitz, I.M. Synthesis of Feedback Systems, Academic Press, Inc., New York, 1963.
8. Rung, B. T., and Thaler, G. J. "Feedback Control Systems: Design with Regard to Sensitivity," IEEE Trans, Appl. & Ind., Vol. 83, No. 75, Nov. 1964, p. 360-366.
9. Pagurek, B. "Sensitivity of the Performance of Optimal Control Systems to Plant Parameter Variations," IEEE Trans. on Automatic Control, April 1965, p. 178-180.
10. Zadeh, L. A., and Desoer, C. A. Linear System Theory, McGraw-Hill Book Co., Inc., New York, 1963.
11. Laughton, M. A. "Sensitivity in Dynamic System Analysis," J. Elec. & Control, Vol. 17, 1964, p. 577-591.

12. Pagurek, B. "Sensitivity of the Performance of Optimal Linear Control Systems to Parameter Variations," Int. J. of Control, Vol. 1, 1965, p. 33-45.
13. Cruz, J. B., and Perkins, W. R. "A New Approach to the Sensitivity Problem in Multivariable Feedback System Design," IEEE Trans. on Automatic Control, July 1964, p. 216-223.
14. Lindorff, D. L. "Sensitivity in Sampled Data Systems," IRE Trans. on Automatic Control, April 1963, p. 120-125.
15. Horowitz, I. M. "Fundamental Theory of Automatic Linear Feedback Control Systems," IRE Trans. on Automatic Control, Dec. 1959, p. 5-19.
16. Gardner, M. F., and Barnes, J. L. Transients in Linear Systems, John Wiley & Sons, Inc., New York, 1942.
17. Greensite, A. Design Criteria for Control of Space Vehicles; Vol. II, part 1, Linear Systems, General Dynamics Convair Report GDC-DDE66-019, 25 April 1966.
18. Frazer, R. A., Duncan, W. J., and Collar, A. R. Elementary Matrices, Cambridge University Press, London, 1938.
19. Horowitz, I. M. "Plant Adaptive Systems Versus Ordinary Feedback Systems," IRE Trans. on Automatic Control, Jan. 1962, p. 48-56.
20. Horowitz, I. M. "Linear Adaptive Flight Control Design For Re-Entry Vehicles," IEEE Trans. on Automatic Control, Jan. 1964, p. 90-97.
21. LaBounty, R. H., and Houppis, C. H. "Root Locus Analysis of a High Gain Linear System with Variable Coefficients: Application of Horowitz's Method," IEEE Trans. on Automatic Control, April 1966, p. 255-263.
22. Margolis, M., and Leondes, C. T. "On the Philosophy of Adaptive Control for Process Adaptive Systems," Proc. NEC, Vol. 15, 1959, p. 27-33.

23. Kalman, R. E. "Design of a Self-optimizing Control System," Trans. ASME, Vol. 80, No. 2, Feb. 1958, p. 468-478.
24. Staffin, R., and Truxal, J. G. Executive-controlled Adaptive Systems, Research Report R-688-58, PIB-616, Microwave Research Institute, Brooklyn, N. Y., Sept. 1958.
25. Anderson, G. W., et al. "A Self-adjusting System for Optimum Dynamic Performance," IRE Nat. Convention Record, part 4, 1958, p. 182-190.
26. Mishkin, E., and Haddak, R. A. "Identification and Command Problems in Adaptive Systems," IRE WESCON Convention Record, part 4, 1959, p. 125-135.
27. Gibson, J. E., and McVey, E. S. "Multidimensional Adaptive Control," Proc. NEC, Vol. 15, 1959, p. 17-26.
28. Rohrer, R. A., and Sobral, M. "Sensitivity Considerations in Optimal System Design," IEEE Trans. on Automatic Control, Jan. 1965, p. 43-48.
29. Dorato, P. "On Sensitivity in Optimal Control Systems," IEEE Trans. on Automatic Control, July 1963, p. 256-257.
30. Kalman, R. E. The Theory of Optimal Control and the Calculus of Variations, Report TR61-3, Research Institute for Advanced Studies, Baltimore, Md., 1961.
31. Greensite, A. Design Criteria for Control of Space Vehicles; Vol. III, part 7, Optimization Methods, General Dynamics Convair Report DDE66-026, 5 July 1966.
32. Tomovic, R. Sensitivity Analysis of Dynamic Systems, McGraw-Hill Book Co., New York, 1963.



## APPENDIX

### LINEAR SYSTEM RESPONSE IN TERMS OF EIGENVECTORS

Consider the unforced dynamic system described by

$$\dot{x} = Ax \tag{A1}$$

$$x(0) = c \tag{A2}$$

where  $x$  is the state vector ( $n \times 1$  matrix),  $A$  is a constant  $n \times n$  matrix, and  $c$  is the initial condition vector.

If the  $i^{\text{th}}$  eigenvalue of  $A$  is denoted by  $\lambda_i$  (assumed distinct) and the corresponding eigenvector by  $u_i$ , then

$$A u_i = \lambda_i u_i \tag{A3}$$

$$i = 1, 2, \dots, n$$

and

$$u_i^T u_j = u_j^T u_i = 0 \quad \text{whenever } i \neq j \tag{A4}$$

Eqs. (A3) and (A4) are well known properties of eigenvectors.

The modal matrix for  $A$  may be written as

$$M = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}$$

After calculating the inverse, we express it in the form

$$M^{-1} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

where  $v_i$  is a row vector ( $1 \times n$  matrix). By virtue of the relation

$$M^{-1}M = I$$

it follows that †

$$v_i u_j = \delta_{ij} \tag{A5}$$

where  $\delta_{ij}$  is the Dirac delta function.

Now any vector,  $x$ , can be represented as

$$x = \sum_{i=1}^n \xi_i u_i \tag{A6}$$

where the  $\xi_i$  are appropriate scalars.

Premultiplying (A6) by  $v_k$ ,

$$v_k x = v_k \sum_{i=1}^n \xi_i v_k u_i$$

and summing over all  $k$ ,

$$\sum_{k=1}^n v_k x = \sum_{i=1}^n \xi_i \sum_{k=1}^n v_k u_i = \sum_{i=1}^n \xi_i = \sum_{k=1}^n \xi_k$$

by virtue of (A5).

It follows that

$$\xi_k = v_k x \tag{A7}$$

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† In the language of formal matrix theory, the set of vectors  $\{u_i\}$  is said to form a basis, and the set of vectors  $\{v_j\}$  is the reciprocal basis.

and therefore

$$\mathbf{x} = \sum_{i=1}^n v_i \mathbf{x} u_i \quad (\text{A8})$$

Using this result, we may write

$$A \mathbf{x} = A \sum_{i=1}^n v_i \mathbf{x} u_i$$

But since  $(v_i \mathbf{x})$  is a scalar, we may transpose to

$$A \mathbf{x} = \sum_{i=1}^n (v_i \mathbf{x}) A u_i = \sum_{i=1}^n (v_i \mathbf{x}) \lambda_i u_i$$

after applying (A3). Finally,

$$A \mathbf{x} = \sum_{i=1}^n \lambda_i u_i v_i \mathbf{x}$$

which leads to

$$A = \sum_{i=1}^n \lambda_i u_i v_i \quad (\text{A9})$$

Now the solution of Eq. (A1) may be written in the form

$$\mathbf{x} = \sum_{i=1}^n \alpha_i(t) u_i \quad (\text{A10})$$

where the  $\alpha_i(t)$  are scalar functions of time to be determined. In view of Eqs. (A2) and (A10), we have

$$\mathbf{c} = \sum_{i=1}^n \alpha_i(0) u_i$$

or

$$\alpha_i(0) = v_i \mathbf{c} \quad (\text{A11})$$

using (A7).

Substituting Eq. (A10) in (A1), we find

$$\sum_{i=1}^n \dot{\alpha}_i u_i = A \sum_{i=1}^n \alpha_i u_i = \sum_{i=1}^n \alpha_i A u_i = \sum_{i=1}^n \alpha_i \tau_i u_i$$

using (A3).

Therefore

$$\dot{\alpha}_i = \lambda_i \alpha_i \tag{A12}$$

The solution of this equation, subject to the initial condition (A11), is

$$\alpha_i(t) = e^{\lambda_i t} v_i c \tag{A13}$$

Consequently, after substituting Eq. (A13) in (A10), we obtain

$$x = \sum_{i=1}^n e^{\lambda_i t} v_i c u_i$$

or, equivalently,

$$x = \sum_{i=1}^n e^{\lambda_i t} u_i v_i c \tag{A14}$$

This is the result sought. It expresses the response of the system as a weighted sum of the individual modes.

Since the solution to Eq. (A1) may be expressed in the equivalent form

$$x = e^{At} c \tag{A15}$$

a comparison of (A14) and A15) shows that

$$e^{At} = \sum_{i=1}^n e^{\lambda_i t} u_i v_i \tag{A16}$$